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The Golod–Shafarevich inequality and the class field tower problem

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Resum *(CAT)*
En aquest article presentem una demostració del problema de la torre de cossos de classes. Comencem introduint els grups pro-p, expliquem com descriure'ls en termes de generadors i relacions, i presentem la desigualtat de Golod–Shafarevich, la qual estableix un criteri perquè un grup pro-p sigui infinit. Després d'introduir algunes nocions de teoria algebraica de nombres, apliquem la desigualtat de Golod–Shafarevich al problema de la torre de cossos de classes. Obtenim un criteri perquè un cos de nombres tingui una torre de cossos de classes infinita, i donem exemples explícits de cossos de nombres satisfent aquest criteri.

Abstract *(ENG)*
In this article we present a proof of the class field tower problem. We begin by introducing pro-p groups, explain how to describe them in terms of generators and relations, and present the Golod–Shafarevich inequality, which establishes a criterion for a pro-p group to be infinite. After introducing some notions from algebraic number theory, we apply the Golod–Shafarevich inequality to the class field tower problem. We obtain a criterion for a number field to have an infinite class field tower, and give explicit examples of number fields satisfying this criterion.

Keywords: *number theory, class field tower problem, pro-p groups, Golod–Shafarevich inequality.*
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1. Introduction

During the 19th century, class field theory developed around three main themes: relations between abelian extensions and ideal class groups, density theorems for primes using L-functions, and reciprocity laws. As explained in [6], the need to study class field towers originated with the only conjecture of Hilbert concerning the Hilbert class field which turned out to be incorrect, namely the claim that the Hilbert class field of a number field with class number 4 has odd class number.

In 1916, Philipp Furtwängler realized that the Hilbert 2-class field $\mathbb{H}_{(2)}(K)$ of a number field K with 2-class group isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ need not have odd class number. He observed that Hilbert's method to prove the quadratic reciprocity law in K would still work if the 2-class field $\mathbb{H}^2_{(2)}(K)$ of $\mathbb{H}_{(2)}(K)$ had odd class number. This made Furtwängler ask the following question: does the p-class field tower of a number field K always terminate?

A negative answer to that question would solve the class field tower problem, which asks whether the class field tower of any number field always terminates. This problem was posed by Furtwängler in 1925 and remained open for almost 40 years, with no clear indication whether the answer should be positive or negative. By class field theory, this problem is equivalent to the following question: *Given a number field K, does it always exist a finite extension L of K such that the ring of integers of L is a principal ideal domain?*.

The class field tower problem could be solved by finding a number field K whose maximal unramified prosolvable extension has infinite degree over K. A convenient way to construct such K would be to prove that for some prime p, the maximal unramified pro-p extension $\mathbb{H}^\infty_p(K)$ of K has infinite degree, or equivalently, that the Galois group $G_{K,p} := \mathrm{Gal}(\mathbb{H}^\infty_p(K)/K)$ is infinite.

A major evidence for the negative answer to the class field tower problem was given by Igor Shafarevich in 1963 (see [9]), where the formula for the minimal number of generators $d(G_{K,p})$ of $G_{K,p}$ and an upper bound for the minimal number of relations $r(G_{K,p})$ were established. A year later, in 1964, Golod and Shafarevich (see [3]) were able to produce counterexamples for the p-class field tower problem by showing that for any finite p-group G, the minimal numbers of generators $d(G)$ and relations $r(G)$ (where G is considered as a pro-p group) are related by the inequality $r(G) > (d(G)-1)^2/4$. This was improved to $r(G) > d(G)^2/4$ in the subsequent works of Vinberg (see [10]) and Roquette (see [8]). This inequality is known as the Golod–Shafarevich inequality. Golod and Shafarevich applied this inequality to $G_{K,p}$, that is by definition a pro-p group, and use this to obtain a criterion for the p-class field tower of K to be infinite.

The aim of this article is to present a proof of the class field tower problem, as well as provide the necessary framework to be able to formulate this problem and solve it. We begin with a brief introduction to pro-p groups that lead to the formulation of the Golod–Shafarevich inequality, following [5] as the main reference. We then introduce some notions from algebraic number theory and class field theory, based on [7], [4] and [1]. We conclude by explaining the solution to the class field tower problem, giving some particular counterexamples of number fields with an infinite class field tower. Most of the results in this last part are taken from [2].

2. The Golod–Shafarevich inequality

Definition 2.1. A *profinite group* is a topological group that can be realized as a projective limit of discrete finite groups.

These groups have an important role in number theory, as Galois groups of algebraic field extensions are always profinite. We are interested in a particular type of profinite groups, called *pro-p groups*, which are those profinite groups that can be realized as an inverse limit of finite p-groups. These groups describe the Galois groups of p-extensions.

Definition 2.2. Let G be a pro-p group. A *system of generators* of G is a subset $E \subseteq G$ with the following properties:

(i) G is the smallest closed subgroup containing E,

(ii) every neighborhood of $1 \in G$ contains all but finitely many elements of E.

We say E is *minimal* if no proper subset of E is a system of generators of G.

As when working with regular groups, we can define an analog of a free group and express a pro-p group in terms of generators and relations.

Definition 2.3. Let I be an index set and let F_I be the free group with generators $\{s_i \mid i \in I\}$. Let $\mathfrak{U}$ be the set of all normal subgroups N of F_I satisfying that

(i) $[F_I : N]$ is a power of p,

(ii) almost all elements of $\{s_i \mid i \in I\}$ are in N.

We define the *free pro-p group with system of generators* $\{s_i \mid i \in I\}$ as

$$F(I) := \varprojlim_{N \in \mathfrak{U}} F_I/N.$$

The group F_I embeds into $F(I)$ by $g \mapsto \prod gN$, and the image of F_I is dense in $F(I)$. Through this embedding, the set $\{s_i \mid i \in I\}$ is in fact a minimal system of generators of $F(I)$.

Example 2.4. Let $I = \{1\}$. Then $F_I = \mathbb{Z}$ and the subgroups $N \in \mathfrak{U}$ in this case are precisely the subgroups $\mathbb{Z}/p^n\mathbb{Z}$. Thus, the free pro-p group generated by a singleton is

$$F(\{1\}) = \varprojlim_{n \geq 0} \mathbb{Z}/p^n\mathbb{Z} = \mathbb{Z}_p.$$

We say that 1 is a topological generator of $\mathbb{Z}_p$, since $\{1\}$ is a system of generators of this pro-p group as defined above. Nevertheless, observe that 1 does not generate $\mathbb{Z}_p$ as a group.

Definition 2.5. Let G be a pro-p group and let $F(I)$ be a free pro-p group with system of generators $\{s_i \mid i \in I\}$. A *presentation* of G by $F(I)$ is an exact sequence of of pro-p groups

$$1 \longrightarrow R \longrightarrow F(I) \xrightarrow{\varphi} G \longrightarrow 1.$$

We identify R with the corresponding subgroup of F. If $\{\varphi(s_i) \mid i \in I\}$ is a minimal system of generators of G, then the presentation is called *minimal*.

Definition 2.6. Given a presentation of G as in the previous definition, a subset $E \subseteq R$ is called a *system of relations* of G if it satisfies:

(i) R is the smallest normal subgroup of F containing E,

(ii) every open normal subgroup of R contains almost all elements of E.

We say that E is *minimal* if no proper subset of E is a system of relations of G.

The first and second cohomology groups of a pro-p group G play a very important role since they allow us to define two very important invariants. If we consider the trivial action of G on $\mathbb{F}_p$, we can regard the cohomology groups $H_n(G, \mathbb{F}_p)$ as $\mathbb{F}_p$-vector spaces. Then, we define the *generator rank* of G as $d(G) := \dim_{\mathbb{F}_p}(H_1(G, \mathbb{F}_p))$ and the the *relation rank* of G as $r(G) := \dim_{\mathbb{F}_p}(H_2(G, \mathbb{F}_p))$. The name given to these invariants is justified by the following theorem:

Theorem 2.7. *The generator rank of a pro-p group G equals the cardinality of any minimal system of generators, and the relation rank equals the cardinality of any minimal system of relations.*

Observation 2.8. The previous theorem tells us, in particular, that any two minimal systems of generators have the same cardinality, and so do any two minimal system of relations. Moreover, this last number is independent of the chosen minimal presentation of G.

One would expect that if the generator rank of G is large compared to the relation rank, then G is infinite. Indeed, the following theorem establishes a sufficient condition for this to happen:

Theorem 2.9 (Golod–Shafarevich inequality)**.** *Let G be a finitely generated pro-p group with $d(G) > 1$. If*

$$\frac{d(G)^2}{4} > r(G),$$

then G is infinite.

3. Results from algebraic number theory

3.1 Places of a number field and ramification

Let K be a number field. We denote by $\mathcal{O}_K$ its ring of integers. For every nonzero prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ and any real constant $c \in (0,1)$, the function $|\alpha|_{\mathfrak{p}} := c^{\mathrm{ord}_{\mathfrak{p}}(\alpha)}$ for $\alpha \in K^*$ (and $|0|_{\mathfrak{p}} = 0$) defines a non-Archimedean absolute value on K. We call this a $\mathfrak{p}$*-adic absolute value.* For any two different prime ideals $\mathfrak{p}$ and $\mathfrak{q}$, a $\mathfrak{p}$-adic and a $\mathfrak{q}$-adic absolute values are inequivalent, i.e., they generate different topologies.

On the other side, any embedding σ of K into $\mathbb{R}$ or $\mathbb{C}$ give rise to an Archimedean absolute value by setting $|\alpha|_\sigma = |\sigma(\alpha)|$, where $|\cdot|$ is the usual absolute value on $\mathbb{R}$ or $\mathbb{C}$. Two embeddings give rise to equivalent absolute values if, and only if, they are complex conjugates.

Ostrowski's theorem tells us that any nontrivial absolute value on K is equivalent to a $\mathfrak{p}$-adic absolute value or to an absolute value coming from a real or complex embedding of K. An equivalence class of nontrivial absolute values on K is called a *place* of K. By tradition, a place is called an *infinite place* if it contains an Archimedean absolute value, and a *finite place* otherwise. We shall now describe how places split when extended to a finite extension L of K. Let's begin with finite places.

Every finite place of K can be uniquely identified with a nonzero prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$. We can describe how a place splits when extended in L by describing how $\mathfrak{p}$ splits when extended in $\mathcal{O}_L$. From now on, the term "prime ideal" will be used to mean "nonzero prime ideal".

Fix a prime ideal $\mathfrak{p}$ of $\mathcal{O}_k$. We denote by $\mathfrak{p}O_L$ the ideal generated by $\mathfrak{p}$ in $\mathcal{O}_L$. If a prime ideal $\mathfrak{P}$ of $\mathcal{O}_L$ divides $\mathfrak{p}\mathcal{O}_L$, we say that $\mathfrak{P}$ *lies over* $\mathfrak{p}$ or that $\mathfrak{p}$ *lies under* $\mathfrak{P}$. Every prime ideal of $\mathcal{O}_L$ lies over a unique prime ideal of $\mathcal{O}_K$ and every prime ideal of $\mathcal{O}_K$ lies under at least one prime ideal of $\mathcal{O}_L$.

The primes lying over $\mathfrak{p}$ are exactly the ones which occur in the prime decomposition of $\mathfrak{p}\mathcal{O}_L$. The exponent with which they occur are called the *ramification indices*. Thus, if $\mathfrak{P}^e$ is the exact power of $\mathfrak{P}$ dividing $\mathfrak{p}\mathcal{O}_L$, then e is the ramification index of $\mathfrak{P}$ over $\mathfrak{p}$, denoted by $e(\mathfrak{P}|\mathfrak{p})$. We say that $\mathfrak{p}$ is *unramified* if $e(\mathfrak{P}|\mathfrak{p}) = 1$ for all prime ideals $\mathfrak{P}$ of $\mathcal{O}_L$ lying over $\mathfrak{p}$, and *ramified* otherwise.

If $\mathfrak{P}$ is a prime ideal of $\mathcal{O}_L$ lying over $\mathfrak{p}$, the residue field $\mathcal{O}_K/\mathfrak{p}$ is canonically embedded into the residue field $\mathcal{O}_L/\mathfrak{P}$. The degree of this extension is called the *inertial degree* of $\mathfrak{P}$ over $\mathfrak{p}$, and it is denoted by $f(\mathfrak{P}|\mathfrak{p})$. The inertial degree is always finite, since it is bounded by $[L : K]$.

Let's now describe how infinite places split when extended in a finite extension L of K. An infinite place ν of K is called a *real place* if the completion of K with respect to any absolute value contained in ν is $\mathbb{R}$. Similarly, ν is called a *complex place* if the completion of K with respect to any absolute value contained in ν is $\mathbb{C}$. Thus, the real places of K correspond to the distinct embeddings of K into $\mathbb{R}$ and the complex places correspond to the conjugate pairs of embeddings of K into $\mathbb{C}$. We will describe how ν splits when extended in L by describing how its corresponding embedding can be extended to different embeddings of L into $\mathbb{R}$ or $\mathbb{C}$.

Consider first that ν is a complex place of K and let $\sigma\colon K \hookrightarrow \mathbb{C}$ be an embedding of K into $\mathbb{C}$ such that $|\sigma(x)|$ is in ν. As $\mathbb{C}$ is algebraically closed, we know from Galois theory that there are exactly $n = [L : K]$ different embeddings $\sigma_i\colon L \hookrightarrow \mathbb{C}$ such that $\sigma_i|_K = \sigma$. No two σ_i can be conjugates, as then they would not agree on K. Hence, they represent n distinct complex infinite places $\omega_1, \dots, \omega_n$ of L. We can write

$$\nu = \omega_1 \cdots \omega_n$$

to indicate that the ω_i are the places of L extending ν. In this case, we define the ramification indices $e(\omega_i|\nu)$ and the inertial degrees $f(\omega_i|\nu)$ to be one, and we say that the complex place ν is unramified in L.

Consider now that ν is a real place of K and let $\sigma\colon K \hookrightarrow \mathbb{R}$ be the corresponding embedding. Regarding σ as an embedding from K into $\mathbb{C}$, we can apply Galois theory again to assure the existence of exactly $n = [L : K]$ different extensions of σ to L, some of which may have an image inside $\mathbb{R}$. List the extensions of σ as

$$\sigma_1, \dots, \sigma_r, \sigma_{r+1}, \overline{\sigma}_{r+1}, \dots, \sigma_{r+s}, \overline{\sigma}_{r+s},$$

where $\sigma_i(L) \subset \mathbb{R}$ for $1 \leq i \leq r$ and σ_{r+j}, $\overline{\sigma}_{r+j}$ give s pairs of complex conjugate embeddings of L into $\mathbb{C}$. Note that $r + 2s = n$. This give rise to r distinct real places $\omega_1, \dots, \omega_r$ and s distinct complex places $\omega_{r+1}, \dots, \omega_{r+s}$ of L extending ν. We define the ramification indices as follows: if ω_i is a real place of L lying over ν, we set $e(\omega_i|\nu) = 1$. If ω_{r+j} is a complex place, we set $e(\omega_{r+j}|\nu) = 2$. We define all inertial degrees to be one. Thus, we formally write

$$\nu = \omega_1 \cdots \omega_r \omega_{r+1}^2 \cdots \omega_{r+s}^2.$$

Definition 3.1. We say that an extension of number fields L/K is *unramified* if every place of K (finite and infinite) is unramified in L. More generally, if S is a set of places of K, we say that L/K is unramified outside S if all places of K not belonging to S are unramified in L.

Galois theory can be applied to the general problem of determining how places of a number field split in an extension field, as there are connections between the ramification indices and the inertial degrees introduced before with some subgroups of the Galois group of a given extension. The following theorem tells us that unramified Galois extensions remain unramified after lifting:

Theorem 3.2. *Let L/K be an unramified Galois extension of number fields and F a finite extension of K. Then LF/F is unramified.*

A similar property holds for the compositum of unramified Galois extensions:

Theorem 3.3. *Let L/K and F/K be Galois extensions of number fields. Let S be a set of places of K. Suppose L/K and F/K are unramified outside S. Then, LF/K is also unramified outside S.*

Applying this theorem to $S = \emptyset$ we obtain the following result:

Corollary 3.4. *Let L/K and F/K be unramified Galois extensions of number fields. Then, LF/K is unramified.*

3.2 The Hilbert class field

For a number field K, we denote by $\mathrm{Cl}(K)$ the *class group* of K, i.e., the quotient group of the fractional ideals of $\mathcal{O}_K$ by its subgroup of principal ideals. Its cardinality is known as the *class number* of K, and it is always finite.

In 1898, Hilbert stated the following conjecture:

Conjecture 3.5. *For any number field K there is a unique finite extension L such that*

(i) *L/K is Galois and $\mathrm{Gal}(L/K) \cong \mathrm{Cl}(K)$.*

(ii) *L/K is unramified, and every abelian unramified extension of K is a subfield of L.*

(iii) *For every finite place $\mathfrak{p}$ of K, the inertial degree $f(\mathfrak{P}|\mathfrak{p})$ (for any place $\mathfrak{P}$ of L lying over $\mathfrak{p}$) is the order of $\mathfrak{p}$ in $\mathrm{Cl}(K)$.*

(iv) *Every ideal of $\mathcal{O}_K$ becomes principal in $\mathcal{O}_L$.*

Hilbert proved the existence of such extension when the class number was 2 and $[K : \mathbb{Q}] = 2$. In 1907, Philipp Furtwängler proved the first two parts of Hilbert's conjecture in general, and used this to prove the quadratic reciprocity law in all number fields in 1913. He proved the third part in 1911 and the fourth part in 1930, after Artin reduced it to a purely group-theoretic statement.

Property (ii) is normally used to characterize this extension:

Definition 3.6. Let K be a number field. The *Hilbert class field* of K, denoted by $\mathbb{H}(K)$, is the maximal unramified abelian extension of K.

4. The class field tower problem

We begin by formulating the following problem:

Problem 4.1 (Embeddability problem)**.** *Given a number field K, does it always exist a finite extension L of K such that the $\mathcal{O}_L$ is a principal ideal domain?*

If K is a number field, the extent to which $\mathcal{O}_K$ fails to be a PID is measured by the class group $\mathrm{Cl}(K)$. In particular, $\mathcal{O}_K$ is a PID if, and only if, $\mathrm{Cl}(K)$ is trivial. As explained in the previous section, the class group of K is isomorphic to the Galois group $\mathrm{Gal}(\mathbb{H}(K)/K)$. Thus, $\mathcal{O}_K$ is a PID if, and only if, the Hilbert class field of K is K itself. This brings us to consider another problem. To state it, we need the following definition:

Definition 4.2. The *class field tower* of K is the tower of extensions

$$K = \mathbb{H}^0(K) \subseteq \mathbb{H}^1(K) \subseteq \mathbb{H}^2(K) \subseteq \cdots,$$

where $\mathbb{H}^m(K)$ is the Hilbert class field of $\mathbb{H}^{m-1}(K)$. We say that the class field tower is *finite* if it stabilizes at some point.

Problem 4.3 (Class field tower problem)**.** *Is the class field tower of any number field K always finite?*

The two previous problems are equivalent in the following sense:

Lemma 4.4. *Let K be a number field. Then, the class field tower of K is finite if, and only if, there exists a finite extension L/K with $\mathrm{Cl}(L) = \{1\}$.*

Proof. Assume that the class field tower is finite. Then, there exists $m \in \mathbb{N}$ with $\mathbb{H}(\mathbb{H}^m(K)) = \mathbb{H}^m(K)$ and hence $\mathrm{Cl}(\mathbb{H}^m(K)) = \{1\}$. Since the Hilbert class field of any number field is a finite extension of itself, $\mathbb{H}^m(K)/K$ is finite.

Assume now that L is a finite extension of K with trivial class group and consider the tower of extensions

$$L = LK \subseteq L\mathbb{H}^1(K) \subseteq L\mathbb{H}^2(K) \subseteq \cdots,$$

which is obtained by lifting the class field tower of K by L. By Theorem 3.2, $L\mathbb{H}^{n+1}(K)/L\mathbb{H}^n(K)$ is an abelian unramified extension for every $n \in \mathbb{N}$. In particular, $L\mathbb{H}^1(K)$ is an abelian unramified extension of L. But $\mathrm{Cl}(L) = \{1\}$, so $\mathbb{H}(L) = L$ and L does not have nontrivial abelian unramified extensions. This implies that $L\mathbb{H}^1(K) = L$. Repeating this argument inductively we find that $L\mathbb{H}^n(K) = L$ for all $n \geq 0$. Since $\mathbb{H}^n(K) \subseteq L\mathbb{H}^n(K) = L$, every field in the class field tower of K is contained in L. L is a finite extension of K, so the class field tower of K must be finite. □

4.1 A criterion for infinite class field towers

In general, computing the class field of a given number is a rather difficult task. It's a bit easier to work with the p-class field, defined in the following:

Definition 4.5. Let p be a fixed prime number. The p-class field of K, denoted by $\mathbb{H}_p(K)$, is the maximal unramified Galois extension of K such that the Galois group $\mathrm{Gal}(\mathbb{H}_p(K)/K)$ is an elementary abelian p-group, i.e., an abelian group where every nontrivial element has order p.

Analogously to the class field tower, we define the *p-class field tower* of K as the tower

$$K = \mathbb{H}_p^0(K) \subseteq \mathbb{H}_p^1(K) \subseteq \mathbb{H}_p^2(K) \subseteq \cdots$$

One has that $\mathbb{H}_p^n(K) \subseteq \mathbb{H}^n(K)$. Hence, if the p-class field tower of K is infinite for some prime number p, then so is its class field tower. For a given p, consider the following extension of K:

$$\mathbb{H}_p^\infty(K) := \bigcup_{n\geq 0} \mathbb{H}_p^n(K).$$

Clearly, the p-class field tower of K is finite if, and only if, $\mathbb{H}_p^\infty(K)$ is a finite extension of K. Our goal now will be to give sufficient conditions for $\mathbb{H}_p^\infty(K)/K$ to be infinite. The extension $\mathbb{H}_p^\infty(K)/K$ is unramified, since all its finite subextensions are. Moreover, it is Galois with Galois group

$$\mathrm{Gal}(\mathbb{H}_p^\infty(K)/K) = \varprojlim_{n\geq 0} \mathrm{Gal}(\mathbb{H}_p^n(K)/K).$$

As $\mathrm{Gal}(\mathbb{H}_p^n(K)/K)$ are finite p-groups, $\mathrm{Gal}(\mathbb{H}_p^\infty(K)/K)$ is pro-p. In fact, the following theorem holds:

Theorem 4.6. $\mathbb{H}_p^\infty(K)$ *is the maximal unramified pro-p extension of* K.

Let $G_{K,p} := \mathrm{Gal}(\mathbb{H}_p^\infty(K)/K)$. Proving that $\mathbb{H}_p^\infty(K)/K$ is an infinite extension is equivalent to proving that $G_{K,p}$ is infinite. Let $\mathrm{Fr}(G_{K,p})$ be the Frattini subgroup of $G_{K,p}$. Then, the quotient $G_{K,p}/\mathrm{Fr}(G_{K,p})$, known as the Frattini quotient, is isomorphic to $\mathrm{Gal}(\mathbb{H}_p(K)/K)$.

By the definition of the p-class field of K and Galois theory, $\mathrm{Gal}(\mathbb{H}_p(K)/K)$ is the maximal elementary abelian quotient of $\mathrm{Gal}(\mathbb{H}(K)/K)$. The correspondence between subgroups and quotients of a finite abelian group tells us that $\mathrm{Gal}(\mathbb{H}_p(K)/K)$ is isomorphic to the maximal elementary abelian subgroup of $\mathrm{Gal}(\mathbb{H}(K)/K)$, i.e., $\mathrm{Gal}(\mathbb{H}(K)/K)[p]$. Taking into account that $\mathrm{Gal}(\mathbb{H}(K)/K) \cong \mathrm{Cl}(K)$, we obtain that

$$\mathrm{Gal}(\mathbb{H}_p(K)/K) \cong \mathrm{Cl}(K)[p].$$

The generator rank of a pro-p group is the same as the generator rank of its Frattini quotient, and thus

$$d(G_{K,p}) = d(G_{K,p}/\mathrm{Fr}(G_{K,p})) = d(\mathrm{Gal}(\mathbb{H}_p(K)/K)) = \dim_{\mathbb{F}_p}(\mathrm{H}^1(\mathrm{Cl}(K)[p])).$$

Since $\mathrm{Cl}(K)[p]$ is a finite elementary abelian p-group, $\mathrm{H}^1(\mathrm{Cl}(K)[p]) \cong \mathrm{Cl}(K)[p]$. Let $\rho_p(K) := \dim_{\mathbb{F}_p}(\mathrm{Cl}(K)[p])$ be the p-rank of the class group of K. Then,

$$d(G_{K,p}) = \dim_{\mathbb{F}_p}(\mathrm{H}^1(\mathrm{Cl}(K)[p])) = \dim_{\mathbb{F}_p}(\mathrm{Cl}(K)[p]) = \rho_p(K). \tag{1}$$

The following theorem establishes a relation between the generator and relation ranks of $G_{K,p}$ and the number of infinite places of K:

Theorem 4.7 (Shafarevich)**.** *Let K be a number field and $\nu_\infty(K)$ the number of infinite places of K. Then, for any prime number p we have*

$$0 \leq r(G_{K,p}) - d(G_{K,p}) \leq \nu_\infty(K) - 1.$$

Combining Theorem 4.7 with Theorem 2.9, we obtain the following criterion for the group $G_{K,p}$ to be infinite:

Corollary 4.8 (Golod–Shafarevich)**.** *In the notations above, assume that*

$$\rho_p(K) > 2 + 2\sqrt{\nu_\infty(K) + 1}.$$

Then $G_{K,p}$ is infinite.

Proof. By Equation (1), $\rho_p(K) = d(G_{K,p})$. Rearranging the terms and squaring this inequality we obtain that

$$\frac{d(G_{K,p})^2}{4} - d(G_{K,p}) > \nu_\infty(K).$$

Using Theorem 4.7 we deduce that

$$\frac{d(G_{K,p})^2}{4} > r(G_{K,p}) + 1.$$

Hence $d(G_{K,p}) > 1$ and $d(G_{K,p})^2/4 > r(G_{K,p})$. Theorem 2.9 implies the claim. □

4.2 Particular examples

To complete the negative solution to the class field tower problem it suffices to exhibit examples of number fields satisfying the inequality in Corollary 4.8. We will see that for any prime number p and any $n \in \mathbb{N}$, there exist a number field $K = K(p, n)$ such that $[K : \mathbb{Q}] = p$ and $\rho_p(K) \geq n$. Since $\nu_\infty(K) \leq [K : \mathbb{Q}]$ (because K has $[K : \mathbb{Q}]$ different embedding into $\mathbb{C}$), we can choose any $n > 2 + 2\sqrt{p+1}$. Then, $K(p, n)$ will satisfy the inequality in Corollary 4.8 and hence will have an infinite class field tower.

For $p = 2$, take any $n + 1$ distinct prime numbers $q_1, \dots, q_{n+1}$ congruent to 1 modulo 4. Let $K = \mathbb{Q}(\sqrt{q_1 \cdots q_{n+1}})$ and $L = \mathbb{Q}(\sqrt{q_1}, \dots, \sqrt{q_{n+1}})$. Notice that $[K : \mathbb{Q}] = 2$. One could see that the extension L/K is unramified, and hence $L \subseteq \mathbb{H}(K)$. Observe that L is an abelian extension of K with Galois group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$. By the correspondence between subgroups and the quotients of a finite abelian group, $\mathrm{Gal}(\mathbb{H}(K)/K)$ must have a subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$. Thus, $\rho_2(K) = \dim_{\mathbb{F}_2}(\mathrm{Gal}(\mathbb{H}(K)/K)[2]) \geq \dim_{\mathbb{F}_2}((\mathbb{Z}/2\mathbb{Z})^n) = n$ (it can be shown that, in fact, $\rho_2(K) = n$). For any $n \geq 6 > 2 + 2\sqrt{3}$, by Corollary 4.8, K has an infinite class field tower.

For an odd prime p, take any $n + 1$ distinct prime numbers $q_1, \dots, q_{n+1}$ congruent to 1 modulo p. Let $L_i = \mathbb{Q}(\zeta_{q_i})$ be the q_i-th cyclotomic field and let K_i be the unique subfield of L_i that has degree p over $\mathbb{Q}$. Let $L = L_1 \cdots L_{n+1}$ and $M = K_1 \cdots K_{n+1}$. Since $L_i \cap L_j = \mathbb{Q}$ for $i \neq j$, $\mathrm{Gal}(L/\mathbb{Q}) \cong \bigoplus \mathrm{Gal}(L_i/\mathbb{Q})$, and hence $\mathrm{Gal}(M/\mathbb{Q}) \cong \bigoplus \mathrm{Gal}(K_i/\mathbb{Q}) \cong (\mathbb{Z}/p\mathbb{Z})^{n+1}$. Clearly, $\mathrm{Gal}(M/\mathbb{Q})$ has a subgroup of index p which does not contain $\mathrm{Gal}(K_i/\mathbb{Q})$ for any i. The field K fixed by this subgroup has index p over $\mathbb{Q}$ and is not contained in the compositum of any proper subset of $\{K_1, \dots, K_{n+1}\}$. One could see that the fields KK_i are unramified over K, and hence their compositum M is also unramified over K. In addition M/K is abelian with Galois group $\mathrm{Gal}(M/K) \cong (\mathbb{Z}/p\mathbb{Z})^n$. Then, we must have $M \subseteq \mathbb{H}(K)$. Using again the correspondence between quotients and subgroups of a finite abelian group, we deduce that $\mathrm{Gal}(\mathbb{H}(K)/K)$ has a subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^n$. This shows that $\rho_p(K) = \dim_{\mathbb{F}_p}(\mathrm{Cl}(K)[p]) \geq n$ (again, one could show that the equality holds). For any $n > 2 + 2\sqrt{p+1}$, the field K defined above has an infinite p-class field tower and thus cannot be embedded in a greater number field with class number 1.

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Weak convergence of the Lazy Random Walk to the Brownian motion

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Resum *(CAT)*
En aquest article considerem una modificació del passeig aleatori simple, el *Lazy Random Walk*, i construïm una família de processos estocàstics a partir d'aquest procés que convergeix feblement cap a un moviment Brownià estàndard en una dimensió.

Abstract *(ENG)*
In this paper we consider a modification of the simple random walk, the Lazy Random Walk, and construct a family of stochastic processes from the latter that converges weakly to a standard one-dimensional Brownian motion.

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1. Introduction

Brownian motion can be thought, intuitively, as the random motion of particles suspended in a medium (usually liquid or gas) or as well as a stochastic process with "small" and independent displacements which are independent of the position of the particle. A formal definition containing all these features can be given as follows:

Definition 1.1. A stochastic process $\{B_t : t \geq 0\}$ is a standard one-dimensional Brownian motion if:

(i) $B_0 = 0$ almost surely.

(ii) For any $k \in \mathbb{N}$ and any $0 \leq t_1 < \cdots < t_k < \infty$, the random variables $B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_k} - B_{t_{k-1}}$ are independent.

(iii) For any $0 \leq s < t < \infty$, the random variable $B_t - B_s$ is normally distributed with zero mean and $t - s$ variance.

(iv) The sample paths of the process are continuous everywhere.

However, this last definition includes a couple of properties that do not arise trivially from the intuitive point of view. For instance, why should be the displacements normally distributed or why should the sample paths be continuous functions of the time variable? Besides, there is no guarantee that there is such a mathematical object satisfying all those properties at the same time.

Donsker's Invariance Principle allows us to connect the intuition with mathematical formalism and states that, whenever the displacements are small enough (they have finite variance), we can construct a family of stochastic processes converging weakly (or in law) to a stochastic process whose law verifies Definition 1.1. More particularly:

Theorem 1.2 (Donsker's Invariance Principle)**.** *Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of independent and identically distributed random variables with mean $\mu \in \mathbb{R}$ and variance $0 < \sigma^2 < \infty$ and let $\xi_j = X_j - \mu$. Then the random (continuous) functions*

$$Y_t^{(n)} = \frac{1}{\sigma\sqrt{n}} S_{nt}, \quad 0 \leq t \leq 1, \tag{1}$$

where

$$S_t = \sum_{j=1}^{[t]} \xi_j + (t - [t])\xi_{[t]+1}, \quad S_0 = 0,$$

converge weakly to a standard one-dimensional Brownian motion. In other words, if P_n are the laws of the random functions $Y_t^{(n)}$, then there is a probability measure P (the Wiener measure) over the space of real continuous functions on $[0,1]$, $C[0,1]$, fulfilling the properties from Definition 1.1 and such that $P_n(G) \to P(G)$ for any Borel set G of $C[0,1]$ with $P(\partial G) = 0$.

This result, which is also known as the Functional Central Limit Theorem, can be thought of an analogous of the Central Limit Theorem for random functions.

In this paper we will prove this result in the particular case where $\mathbb{P}\{X_1 = -1\} = \mathbb{P}\{X_1 = 1\} = q/2$, $\mathbb{P}\{X_1 = 0\} = 1 - q$ for $q \in (0,1)$, which is the case of a symmetric Lazy Random Walk, using the same techniques described in [1].

2. Preliminaries

Proving weak convergence in $C[0,1]$ or, in general, in any measurable space $(S,\mathcal{S})$, where S is a metric space and $\mathcal{S}$ is its Borel σ-algebra, can be very complicated. In this section we shall provide the tools that will be used to simplify this task. Let us first recall a characterization of the weak convergence of probability measures given in [4]:

Theorem 2.1. *Let $\{P_n\}_{n\in\mathbb{N}}$ and P be probability measures on $(S,\mathcal{S})$. Then the following statements are equivalent:*

(i) *P_n converges weakly to P,*

(ii) *$\int_S f\,dP_n \to \int_S f\,dP$ for any bounded and continuous function $f\colon S\to\mathbb{R}$,*

(iii) *$P_n(G)\to P(G)$ for any $G\in\mathcal{S}$ such that $P(\partial G)=0$.*

The collection of all the measures on $(S,\mathcal{S})$ can be thought of as a function space from the σ-algebra $\mathcal{S}$ to the interval $[0,1]$. Given this interpretation, and as we do with the space of real continuous functions, we can define a notion of relative compactness.

Definition 2.2. An arbitrary family of probability measures Π on $(S,\mathcal{S})$ is said to be relatively compact if for any sequence $\{P_n\}_{n\in\mathbb{N}}\subset\Pi$ exists a probability P on $(S,\mathcal{S})$ (not necessarily in Π) and a subsequence $\{P_{n_i}\}_{i\in\mathbb{N}}$ converging weakly to P.

The following theorem gives us a characterization of the weak convergence which will be crucial when proving the existence and uniqueness of probability measures on metric spaces:

Theorem 2.3. *Let $\{P_n\}_n$ and P be probabilities on $(S,\mathcal{S})$. Then P_n converges weakly to P if, and only if, every subsequence $\{P_{n_i}\}_i$ has a further subsequence $\{P_{n_{i_m}}\}_m$ converging weakly to P when $m\to\infty$.*

Proof. We will only prove the sufficiency, since it is the useful part. If P_n does not converge weakly to P, by Theorem 2.1, there is a bounded and continuous function $f\colon S\to\mathbb{R}$ such that $\int_S f\,dP_n \not\to \int_S f\,dP$, meaning that for some $\varepsilon>0$ and some subsequence $\{P_{n_i}\}_i$ we have $\left|\int_S f\,dP_{n_i} - \int_S f\,dP\right| > \varepsilon$ for all i. This in particular implies that no further subsequence can be weakly convergent to P. □

Continuing with the parallelism with the space of continuous functions, where the Arzelà–Ascoli Theorem gives a characterization of the relative compactness of a family of functions in terms of equicontinuity and pointwise boundedness, there might be a characterization of these relatively compact families of probability measures in terms of the compact sets of our metric space. In this case, the equivalence is given by Prohorov's Theorem.

Definition 2.4. An arbitrary family of probability measures Π on $(S,\mathcal{S})$ is said to be tight if for any $\varepsilon>0$ there is a compact subset K of S such that $P(K)>1-\varepsilon$ for any $P\in\Pi$.

The tightness of a family of probability measures tells us that there is no escape of mass to "infinity". In other words, all the mass is concentrated in our space S and not in any extension of it.

Theorem 2.5 (Prohorov)**.** *If an arbitrary family of probability measures Π on $(S,\mathcal{S})$ is tight, then it is relatively compact. Furthermore, if S is a polish space and Π is relatively compact, then it is tight.*

Two different proofs of this result can be found in [2] and [4].

Let us now see how we will apply these results in the particular case where $S = C := C[0,1]$ with the uniform metric. First of all, recall that C is a complete and separable metric space with this metric, implying that a family of probability measures on $(C, \mathcal{C})$, where $\mathcal{C}$ is the Borel σ-algebra of $C[0,1]$, is relatively compact if, and only if, it is tight.

Definition 2.6. The finite dimensional distributions of a probability measure on $(C, \mathcal{C})$ are the compositions $P\pi^{-1}_{t_1,\dots,t_k} := P \circ \pi^{-1}_{t_1,\dots,t_k}$, where $k \in \mathbb{N}$, $0 \le t_1 < \cdots < t_k \le 1$ and $\pi_{t_1,\dots,t_k}(f) = (f(t_1), \dots, f(t_k))$ for any $f \in C$.

It can be shown (see [2] or [4] and [3]) that if X is a random function (that is, a measurable function from a sample space with a certain σ-algebra to $(C, \mathcal{C})$), then the laws of the random vectors $(X_{t_1}, \dots, X_{t_k})$ (the finite dimensional random vectors of the stochastic process) coincide with the finite dimensional distributions of the law of the random function X and that the finite dimensional distributions of a probability measure determine unequivocally the probability measure. In other words, if P and Q are probabilities over $(C, \mathcal{C})$ such that $P\pi^{-1}_{t_1,\dots,t_k} = Q\pi^{-1}_{t_1,\dots,t_k}$ for any $k \in \mathbb{N}$ and any $0 \le t_1 < \cdots < t_k \le 1$, then $P = Q$.

Theorem 2.7. *If a sequence of probabilities $\{P_n\}_n$ is relatively compact, and if $P_n\pi^{-1}_{t_1,\dots,t_k}$ converges weakly to some probability measure $\mu_{t_1,\dots,t_k}$ on $(\mathbb{R}^k, \mathcal{R}^k)$ (being $\mathcal{R}^k$ the Borel σ-algebra of $\mathbb{R}^k$) for all $k \in \mathbb{N}$ and for any $0 \le t_1 < \cdots < t_k \le 1$, then some probability P on $(C, \mathcal{C})$ satisfies $P\pi^{-1}_{t_1,\dots,t_k} = \mu_{t_1,\dots,t_k}$ for all k and $t_1, \dots, t_k$ and P_n converges weakly to P.*

Proof. Indeed, let $\{P_{n_i}\}_i$ be any subsequence of $\{P_n\}_n$. Then some further subsequence $\{P_{n_{i_m}}\}_m$ converges weakly to a probability P. Now, since $\pi_{t_1,\dots,t_k} \colon C \to \mathbb{R}^k$ is a continuous function on the space C, by Lebesgue's change of variables formula, we have for any bounded and continuous function, $f \colon C \to \mathbb{R}$,

$$\int_{\mathbb{R}^k} f \cdot d(P_{n_{im}}\pi^{-1}_{t_1\cdots t_k}) = \int_C f \circ \pi_{t_1\cdots t_k} \cdot dP_{n_{im}} \xrightarrow{m\to\infty} \int_C f \circ \pi_{t_1\cdots t_k} \cdot dP = \int_{\mathbb{R}^k} f \cdot d(P\pi^{-1}_{t_1\cdots t_k}).$$

Meaning that $P_{n_{im}}\pi^{-1}_{t_1\cdots t_k}$ converges weakly to $P\pi^{-1}_{t_1\cdots t_k}$ for all k and $t_1, \dots, t_k$. By uniqueness of the weak limit of a sequence of probabilities and Theorem 2.3, we have $P\pi^{-1}_{t_1\cdots t_k} = \mu_{t_1,\dots,t_k}$.

To prove the second half of the theorem, if P is the probability measure found in the first half, we have that $P_n\pi^{-1}_{t_1\cdots t_k}$ converges weakly to $P\pi^{-1}_{t_1\cdots t_k}$ for all k and $t_1, \dots, t_k$. However, given that $\{P_n\}_n$ is relatively compact, any subsequence will have a further subsequence $\{P_{n_{im}}\}_m$ converging weakly to some probability Q on $(C, \mathcal{C})$. Again, by Lebesgue's change of variables formula and uniqueness of the limit, we conclude that $Q\pi^{-1}_{t_1\cdots t_k} = P\pi^{-1}_{t_1\cdots t_k}$ for all k and $t_1, \dots, t_k \in [0,1]$, meaning that $Q = P$. Namely, any subsequence has a further subsequence weakly convergent to P and thus, by Theorem 2.3, we see that P_n weakly converges to P. □

This last result tells us that, in order to prove the existence of the Wiener measure/process and convergence towards this stochastic process, one only has to construct a sequence of stochastic processes whose laws are relatively compact (or, by virtue of Prohorov's Theorem, tight) and then check that the finite dimensional vectors converge in law to the desired ones.

In general, it is easier to prove the tightness of a sequence of random functions (namely, the tightness of their laws) rather than its relative compactness. Billingsley's Criterion provides us a tool towards this direction.

Theorem 2.8 (Billingsley's Criterion). *Let $\{X^{(n)}\}_n$ be a sequence of random functions. Then it is tight if the following conditions are fulfilled:*

(i) $\{X_0^{(n)}\}$ *is tight (as a sequence of random variables).*

(ii) *For some $\gamma \geq 0$, $\alpha > 1$ and some continuous, non-decreasing function $F\colon [0,1] \to \mathbb{R}$, we have*

$$\mathbb{P}\{|X_t^{(n)} - X_s^{(n)}| \geq \lambda\} \leq \frac{1}{\lambda^\gamma}|F(t) - F(s)|^\alpha$$

for any $\lambda > 0$, $n \in \mathbb{N}$ and $s, t \in [0,1]$.

Due to Markov's inequality, if we manage to prove that $\mathbb{E}[|X_t^{(n)} - X_s^{(n)}|^\gamma] \leq |F(t) - F(s)|^\alpha$, condition (ii) of Billingsley's Criterion will be fulfilled.

A proof of this result can be found in [2].

3. Tightness and convergence of the finite dimensional distributions

Let us first prove the tightness of the sequence of random functions defined by Equation (1) for the particular case of the symmetric Lazy Random Walk (note that here we have $\mu = 0$ and $\sigma^2 = q$).

Theorem 3.1. *There is a positive constant C such that $\mathbb{E}[(Y_t^{(n)} - Y_s^{(n)})^4] \leq C(t-s)^2$ for any $n \in \mathbb{N}$ and any $s, t \in [0,1]$.*

Proof. If $s = t$, the result is trivial. Without any loss of generality, let us assume that $0 \leq s < t \leq 1$.

Let us first note that we can rewrite $Y_t^{(n)}$ as follows:

$$Y_t^{(n)} = \frac{1}{\sqrt{n}} \int_0^{nt} \theta(x)\, dx, \quad \theta(x) = \sum_{k=1}^{\infty} \frac{X_k}{\sqrt{q}} \mathbb{I}_{[k-1,k)}(x).$$

Indeed, the first integral in $\int_0^{nt} \theta(x)\, dx = \int_0^{[nt]} \theta(x)\, dx + \int_{[nt]}^{nt} \theta(x)\, dx$ reduces to the integral of the sum $\sum_{k=1}^{[nt]} X_k \mathbb{I}_{[k-1,k)}(x)/\sqrt{q}$ due to the contribution of characteristic functions whose interval $[k-1,k)$ lies in the interval $[0,[nt]]$, while in the second integral, this only occurs in the summand whose characteristic function lies in the interval $[[nt], nt]$. Thus,

$$\int_0^{[nt]} \theta(x)\, dx = \sum_{k=1}^{[nt]} \frac{X_k}{\sqrt{q}} \int_0^{[nt]} \mathbb{I}_{[k-1,k)}(x)\, dx = \sum_{k=1}^{[nt]} \frac{X_k}{\sqrt{q}},$$

$$\int_{[nt]}^{nt} \theta(x)\, dx = \frac{X_{[nt]+1}}{\sqrt{q}} \int_{[nt]}^{nt} dx = (nt - [nt]) \frac{X_{[nt]+1}}{\sqrt{q}}.$$

Now, given that $s < t$, we have that $\sqrt{n}(Y_t^{(n)} - Y_s^{(n)}) = \int_{[ns,nt]} \theta(x)\, dx$, from where we get

$$\begin{aligned}(Y_t^{(n)} - Y_s^{(n)})^4 &= \frac{1}{n^2} \int_{[ns,nt]^4} \theta(x_1)\theta(x_2)\theta(x_3)\theta(x_4)\, d^4\mathbf{x} \\ &= \frac{24}{n^2} \int_{[ns,nt]^4} \theta(x_1)\theta(x_2)\theta(x_3)\theta(x_4)\mathbb{I}_{\{x_1\le x_2\le x_3\le x_4\}}(\mathbf{x})\, d^4\mathbf{x},\end{aligned}$$

where $\mathbf{x} = (x_1, x_2, x_3, x_4)$, $d^4\mathbf{x} = dx_1\, dx_2\, dx_3\, dx_4$ and in the last step we have used that the product $\theta(x_1)\theta(x_2)\theta(x_3)\theta(x_4)$ remains invariant under permutations of the variables $x_1, \dots, x_4$ when $\mathbf{x} \in [ns, nt]^4$ to fix an ordering by making use of the indicator function $\mathbb{I}_{\{x_1\le x_2\le x_3\le x_4\}}$.

Given that the sums $\theta(x)$ are always finite for $x \in [ns, nt]$, the function $\theta(x_1)\theta(x_2)\theta(x_3)\theta(x_4)$ is integrable and we can make use of Fubini's Theorem to say that

$$\mathbb{E}[(Y_t^{(n)} - Y_s^{(n)})^4] = \frac{24}{n^2} \int_{[ns,nt]^4} \mathbb{E}[\theta(x_1)\theta(x_2)\theta(x_3)\theta(x_4)\mathbb{I}_{\{x_1\le x_2\le x_3\le x_4\}}(\mathbf{x})]\, d^4\mathbf{x}.$$

On the other hand, we have that

$$\begin{aligned}&\mathbb{E}[\theta(x_1)\theta(x_2)\theta(x_3)\theta(x_4)\mathbb{I}_{\{x_1\le x_2\le x_3\le x_4\}}(\mathbf{x})] \\ &\qquad = \frac{1}{q^2} \sum_{k_1,k_2,k_3,k_4} \mathbb{E}[X_{k_1}X_{k_2}X_{k_3}X_{k_4}]\mathbb{I}_{[k_1-1,k_1)}(x_1)\cdot \ldots \cdot \mathbb{I}_{[k_4-1,k_4)}(x_4)\mathbb{I}_{\{x_1\le\cdots\le x_4\}}(\mathbf{x}) \\ &\qquad = \sum_{j,k} \left[\left(\frac{1}{q} - 1\right)\delta_{jk} + 1\right] \mathbb{I}_{[k-1,k)^2}(x_1, x_2)\cdot \mathbb{I}_{[j-1,j)^2}(x_3, x_4)\mathbb{I}_{\{x_1\le\cdots\le x_4\}}(\mathbf{x}).\end{aligned}$$

Indeed, given that $\mathbb{E}[X_i] = 0$, $\mathbb{E}[X_i^2] = q$, $\mathbb{E}[X_i^4] = q$ and that the random variables $\{X_i\}_i$ are independent, we have that $\mathbb{E}[X_{k_1}X_{k_2}X_{k_3}X_{k_4}] = 0$ if there is a subscript k_j such that $k_j \neq k_i$ for all $i \in \{1, 2, 3, 4\}\backslash\{j\}$, $\mathbb{E}[X_{k_1}X_{k_2}X_{k_3}X_{k_4}] = q^2$ if $k_1 = k_2 \neq k_3 = k_4$ and $\mathbb{E}[X_{k_1}X_{k_2}X_{k_3}X_{k_4}] = q$ if $k_1 = k_2 = k_3 = k_4$. Any other possible cases are discarded due to the presence of the indicator function $\mathbb{I}_{\{x_1\le x_2\le x_3\le x_4\}}(\mathbf{x})$.

Now, making use of the inequality $\mathbb{I}_{\{x_1\le\cdots\le x_4\}}(\mathbf{x}) \le \mathbb{I}_{\{x_1\le x_2\}}(x_1, x_2)\cdot \mathbb{I}_{\{x_3\le x_4\}}(x_3, x_4)$, we see that

$$\begin{aligned}&\mathbb{E}[(Y_t^{(n)} - Y_s^{(n)})^4] \\ &\quad \le \frac{24}{n^2} \int_{[ns,nt]^4} \sum_k \left(\frac{1}{q}-1\right) \mathbb{I}_{[k-1,k)^2}(x_1, x_2)\cdot \mathbb{I}_{[k-1,k)^2}(x_3, x_4)\mathbb{I}_{\{x_1\le x_2\}}(x_1, x_2)\cdot \mathbb{I}_{\{x_3\le x_4\}}(x_3, x_4)\, d^4\mathbf{x} \\ &\qquad + \frac{24}{n^2} \int_{[ns,nt]^4} \sum_{j,k} \mathbb{I}_{[k-1,k)^2}(x_1, x_2)\cdot \mathbb{I}_{[j-1,j)^2}(x_3, x_4)\mathbb{I}_{\{x_1\le x_2\}}(x_1, x_2)\cdot \mathbb{I}_{\{x_3\le x_4\}}(x_3, x_4)\, d^4\mathbf{x}.\end{aligned} \tag{2}$$

Let us focus on the first term of the latter expression. After some simple manipulations we can rewrite this term as follows:

$$\begin{aligned}&\frac{24}{n^2}\left(\frac{1}{q} - 1\right) \sum_k \left(\int_{[ns,nt]^2} \mathbb{I}_{[k-1,k)^2}(x_1, x_2)\cdot \mathbb{I}_{\{x_1\le x_2\}}(x_1, x_2)\, dx_1\, dx_2\right)^2 \\ &\qquad\qquad = 24n^2\left(\frac{1}{q} - 1\right) \sum_k \left(\int_{[s,t]^2} \mathbb{I}_{\left[\frac{k-1}{n},\frac{k}{n}\right)^2}(y_1, y_2)\cdot \mathbb{I}_{\{y_1\le y_2\}}(y_1, y_2)\, dy_1\, dy_2\right)^2,\end{aligned}$$

where in the last step we have introduced the change of variables $y_i = x_i/n$. Now recall that if $\{a_k\}_k$ is a sequence of non-negative real numbers, then $\left(\sum_k a_k\right)^2 = \sum_k a_k^2 + \sum_{k\neq l} a_k a_l \geq \sum_k a_k^2$, from where we infer that the latter expression is lesser than

$$24n^2\left(\frac{1}{q}-1\right)\left(\int_{[s,t]^2}\sum_k\left(\mathbb{I}_{\left[\frac{k-1}{n},\frac{k}{n}\right)^2}(y_1,y_2)\right)\mathbb{I}_{\{y_1\leq y_2\}}(y_1,y_2)\,dy_1\,dy_2\right)^2$$
$$\leq 24n^2\left(\frac{1}{q}-1\right)\left(\int_{[s,t]^2}\mathbb{I}_{\left\{y_2-y_1\leq\frac{1}{n}\right\}}(y_1,y_2)\cdot\mathbb{I}_{\{y_1\leq y_2\}}(y_1,y_2)\,dy_1\,dy_2\right)^2.$$

Where we have used that $\sum_k\left(\mathbb{I}_{\left[\frac{k-1}{n},\frac{k}{n}\right)^2}(y_1,y_2)\right)\leq\mathbb{I}_{\left\{y_2-y_1\leq\frac{1}{n}\right\}}$. Lastly, we have that

$$\int_s^t\int_s^{y_2}\mathbb{I}_{\left\{y_2-y_1\leq\frac{1}{n}\right\}}(y_1,y_2)\,dy_1\,dy_2=\int_s^t\int_{\max\{y_2-1/n,s\}}^{y_2}dy_1\,dy_2\leq\int_s^t\int_{y_2-1/n}^{y_2}dy_1\,dy_2=\frac{t-s}{n},$$

which allows us to conclude that the first term in Equation (2) can be bounded by $24\left(\frac{1}{q}-1\right)(t-s)^2$.

Proceeding in a similar manner, we can see that the second term in Equation (2) can be bounded by $24(t-s)^2$ and, all in all, we conclude that

$$\mathbb{E}[(Y_t^{(n)}-Y_s^{(n)})^4]\leq\frac{24}{q}(t-s)^2. \qquad \square$$

With this (and Billingsley's Criterion) we have verified that the sequence of random functions defined by Equation (1) is tight (the sequence of random variables $Y_0^{(n)}$ is tight since it is identically zero for all n). We now see that the finite dimensional vectors of the sequence converge to the desired ones:

Theorem 3.2. *For any $k\in\mathbb{N}$ and any $0\leq t_1<\cdots<t_k\leq 1$, the random vectors $(Y_{t_1}^{(n)},\dots,Y_{t_k}^{(n)})$ converge in law to the random vectors $(B_{t_1},\dots,B_{t_k})$ when $n\to\infty$ and where the random variables B_{t_j} are normally distributed with null mean and variance t_j (with $B_{t_j}=0$ if $t_j=0$) and are such that $B_{t_1}, B_{t_2}-B_{t_1},\dots,B_{t_k}-B_{t_{k-1}}$ are independent and therefore (due to the change of variables formula) the random variables $B_{t_{j+1}}-B_{t_j}$ are normally distributed with zero mean and variance $t_{j+1}-t_j$.*

Proof. Before starting to prove the statement, lets first recall the following facts:

(i) If $\{Z_n\}_n$ and $\{W_n\}_n$ are sequences of random vectors such that for every $\varepsilon>0$

$$\mathbb{P}\{\|Z_n-Y_n\|>\varepsilon\}\xrightarrow{n\to\infty}0,$$

then, if W_n converges in law to a certain random vector W, then so does Z_n.

(ii) If a sequence of random vectors $\{Z_n\}$ in $\mathbb{R}^k$ converges in law to a certain random vector Z and $h\colon\mathbb{R}^k\to\mathbb{R}^{k'}$ is a continuous function, then $h(Z_n)$ converges in law to $h(Z)$. In addition, if h is invertible and its inverse is continuous, then Z_n converges in law to Z if, and only if, $h(Z_n)$ converges in law to $h(Z)$.

(iii) A sequence of random vectors $\{Z_n\}$ in $\mathbb{R}^k$ with characteristic functions φ_n converges in law to a certain random vector Z with characteristic function φ if, and only if, $\varphi_n(\mathbf{u})\to\varphi(\mathbf{u})$ for every $\mathbf{u}\in\mathbb{R}^k$ when $n\to\infty$.

Now, for every $t \in [0,1]$, given that $nt - [nt] \in [0,1)$, we have that

$$\left| Y_t^{(n)} - \frac{1}{\sqrt{n}} S_{[nt]} \right| \le \frac{|X_{[nt]+1}|}{\sqrt{nq}}.$$

Meaning that, by Chebyshev's inequality and for every $\varepsilon > 0$,

$$\mathbb{P}\left\{ \left| Y_t^{(n)} - \frac{1}{\sqrt{n}} S_{[nt]} \right| > \varepsilon \right\} \le \mathbb{P}\{|X_{[nt]+1}| > \varepsilon\sqrt{nq}\} \le \frac{\mathrm{Var}(X_{[nt]+1})}{\varepsilon^2 nq} = \frac{1}{n\varepsilon^2} \xrightarrow{n\to\infty} 0.$$

Which implies, setting $Y_{n,k} = (Y_{t_1}^{(n)}, \dots, Y_{t_k}^{(n)})$ and $S_{n,k} = (S_{[t_1 n]}, \dots, S_{[t_k n]})$,

$$\mathbb{P}\left\{ \left\| Y_{n,k} - \frac{1}{\sqrt{n}} S_{n,k} \right\| > \varepsilon \right\} = \mathbb{P}\left\{ \left\| Y_{n,k} - \frac{1}{\sqrt{n}} S_{n,k} \right\|^2 > \varepsilon^2 \right\} \le \sum_{j=1}^{k} \mathbb{P}\left\{ \left| Y_{t_j}^{(n)} - \frac{1}{\sqrt{n}} S_{[nt_j]} \right| > \frac{\varepsilon}{\sqrt{k}} \right\}$$

and this last quantity goes to zero as n approaches infinity for every $\varepsilon > 0$.

By virtue of the first two facts mentioned before, we shall prove that the random vectors $S^{(n)} = (S_{[t_1 n]}, S_{[t_2 n]} - S_{[t_1 n]}, \dots, S_{[t_k n]} - S_{[t_{k-1} n]})/\sqrt{n}$ converge in law to a random vector $(B_{t_1}, \dots, B_{t_k} - B_{t_{k-1}})$ whose components verify the desired properties.

Now, if $t_0 = 0 < t_1$ (if $t_1 = 0$, we can omit this step and proceed in a similar way), note that

$$\frac{1}{\sqrt{n}}(S_{[t_{l+1} n]} - S_{[t_l n]}) = \frac{1}{\sqrt{nq}} \sum_{j=[t_l n]+1}^{[t_{l+1} n]} X_j$$

for all $l \in 0, \dots, k-1$. Since the random variables $\{X_l\}_l$ are independent, this means that the components of the vector $S^{(n)}$ are independent and thus, we only need to prove that each component $(S_{[t_{l+1} n]} - S_{[t_l n]})/\sqrt{n}$ converges in law to a normal random variable with null mean and variance $t_{l+1} - t_l$ (recall that the characteristic function of a vector whose components are independent is the product of the characteristic functions of each component and therefore the third fact can be applied). If we manage to prove that

$$\mathbb{P}\left\{ \left| \frac{1}{\sqrt{nq}} \sum_{j=[t_l n]+1}^{[t_{l+1} n]} X_j - \frac{\sqrt{t_{l+1} - t_l}}{\sqrt{q[n(t_{l+1} - t_l)]}} \sum_{j=1}^{[n(t_{l+1} - t_l)]} X_j \right| > \varepsilon \right\} \xrightarrow{n\to\infty} 0 \tag{3}$$

for every $\varepsilon > 0$, then, due to the first fact, we will have proven that the two sums (multiplied by their respective factors) will have the same limit in law. But, due to the Central Limit Theorem, the sum

$$\frac{\sqrt{t_{l+1} - t_l}}{\sqrt{q[n(t_{l+1} - t_l)]}} \sum_{j=1}^{[n(t_{l+1} - t_l)]} X_j$$

converges in law to a normal random variable with zero mean and variance $t_{l+1} - t_l$, concluding the proof.

To verify identity (3), we first assume that $[n(t_{l+1} - t_l)] < [nt_l] + 1$. If this is the case, then the random variables X_j involved in Equation (3) are all independent and, by Chebyshev's inequality,

$$\mathbb{P}\left\{ \left| \frac{1}{\sqrt{nq}} \sum_{j=[t_l n]+1}^{[t_{l+1} n]} X_j - \frac{\sqrt{t_{l+1} - t_l}}{\sqrt{q[n(t_{l+1} - t_l)]}} \sum_{j=1}^{[n(t_{l+1} - t_l)]} X_j \right| > \varepsilon \right\} \le \frac{1}{\varepsilon^2} \left[\frac{[nt_{l+1}] - [nt_l]}{n} - (t_{l+1} - t_l) \right].$$

Using that $\lim_{n\to\infty} n \cdot s/[n \cdot s] = 1$ for all fixed $s > 0$, we see that this last quantity goes to zero as n approaches infinity for every $\varepsilon > 0$.

If $[n(t_{l+1} - t_l)] \geq [nt_l] + 1$, we first rewrite the difference in the probability (3) as follows:

$$\frac{1}{\sqrt{nq}} \sum_{j=[t_l n]+1}^{[t_{l+1}n]} X_j - \frac{\sqrt{t_{l+1} - t_l}}{\sqrt{q[n(t_{l+1} - t_l)]}} \sum_{j=1}^{[n(t_{l+1}-t_l)]} X_j$$
$$= \left(\frac{1}{\sqrt{nq}} - \frac{\sqrt{t_{l+1} - t_l}}{\sqrt{q[n(t_{l+1} - t_l)]}} \right) \sum_{j=[nt_l]+1}^{[n(t_{l+1}-t_l)]} X_j + \frac{1}{\sqrt{nq}} \sum_{j=[n(t_{l+1}-t_l)]+1}^{[nt_{l+1}]} X_j - \frac{\sqrt{t_{l+1} - t_l}}{\sqrt{q[n(t_{l+1} - t_l)]}} \sum_{j=1}^{[nt_l]} X_j.$$

Again, the random variables X_j involved are independent and thus, by Chebyshev's inequality,

$$\mathbb{P}\left\{ \left| \frac{1}{\sqrt{nq}} \sum_{j=[t_l n]+1}^{[t_{l+1}n]} X_j - \frac{\sqrt{t_{l+1} - t_l}}{\sqrt{q[n(t_{l+1} - t_l)]}} \sum_{j=1}^{[n(t_{l+1}-t_l)]} X_j \right| > \varepsilon \right\}$$
$$\leq \frac{1}{\varepsilon^2} \left[\left(\frac{1}{\sqrt{n}} - \frac{\sqrt{t_{l+1} - t_l}}{\sqrt{[n(t_{l+1} - t_l)]}} \right)^2 ([n(t_{l+1} - t_l)] - [nt_l]) + \frac{[nt_{l+1}] - [n(t_{l+1} - t_l)]}{n} - \frac{[nt_l](t_{l+1} - t_l)}{[n(t_{l+1} - t_l)]} \right].$$

And this last quantity tends to zero as n tends to infinity for every $\varepsilon > 0$. □

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On the relationship between singularity exponents and finite time Lyapunov exponents in remote sensed images of the ocean

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Resum *(CAT)*
Els processos de transport i mescla horitzontals són clau per descriure la majoria de fenòmens a l'oceà. Les estructures lagrangianes coherents es defineixen com màxims locals dels exponents de Lyapunov finits i expliquen aquests processos. Cal però, una seqüència del camp de velocitats per estimar-los numèricament. Aquí, estudiem fins a quin punt aquests exponents es poden estimar mitjançant l'anàlisi singular de diferents imatges satèl·lit de l'oceà. L'anàlisi singular es basa en la descomposició d'un senyal en components fractals caracteritzats pels anomenats exponents de singularitat.

Abstract *(ENG)*
Horizontal transport and mixing are key to properly understanding changes in the global ocean. Lagrangian Coherent Structures explain those processes and are defined as the local maxima of Finite Size Lyapunov Exponents which can only be estimated by a long enough sequence of the velocity field. We discuss to which extend the exponents can be estimated by using only singularity analysis of remote sensed images of the ocean. Singularity analysis is based on the decomposition of a signal in fractal components characterised by the Singularity Exponents which we compare to the Lyapunov Exponents.

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1. Introduction

Generally speaking, the ocean flows can either be described as laminar or turbulent. Motion in laminar circulation can be well characterized since neighbour particles advected by these flows track similar paths. In the turbulent case, movements are dominated by twirls, eddies and certain randomness. Particles that are close to each other might be widely separated later. Ocean circulation is described by the latter case.

Finite size Lyapunov exponents (FSLE) characterize the rate of separation of close trajectories and therefore, provide information of the dispersion processes and the Lagrangian coherent structures. The information provided by FSLE is enough to assess a major part of the circulation on the global ocean. The main disadvantage of using FLSE to asses dynamic information is that their estimation requires a long sequence of velocity field data. The main objective of this work is to understand the extent to which we can estimate FSLE by using only remote sensed images (without requiring the velocity field). For this, we will explore the functional relationship between the singularity exponents (SE) of the sea surface temperature (SST) signal and the FSLE. The former exponents represent the Eulerian description of dynamics while the latter represent the Lagrangian approach. In general, an autonomous system, with constant velocity fields presents an evident correspondence between Eulerian and Lagrangian descriptions. The same is not evident for turbulent systems.

The article is structured as follows. The first section includes the theoretical background of the Eulerian and the Lagrangian approaches. We provide the mathematical definition of the singularity and Lyapunov exponents, a discussion on how these exponents can describe the underlying dynamics and the introduction of a discrete and finite method to estimate them. The remaining sections are devoted to assess a preliminary functional relation to explain FSLE in terms of SST singularity exponents (SST SE). We compute the SE of sea surface temperature (SST) and absolute dynamic topography (ADT) following an algorithm based on the most unpredictable measure. FSLE are acquired from an external source. Finally, we asses a functional relationship between the exponents. We also evaluate the correspondence between SST SE and ADT SE which are theoretically equivalent.

2. Theoretical background

The rapid rotation of the Earth and the strong stratification of the essentially incompressible[1] water in the ocean generate turbulent dynamics in the open ocean. Indeed, the turbulence is not bidimensional but 3-D in the ocean. Satellite measurements can only capture the 2-D turbulence, but the ocean is 3-D turbulent. We could define two dimensional stream functions that approximate the motion in the ocean, although not always with the required accuracy. In fact, the dynamics in the ocean are characterized by having structures and areas where infinitesimal perturbations grow exponentially in time and thus it defines a chaotic system. Sections 2.1 and 2.2 respectively present the mathematical Lagrangian and Eulerian characterizations of the geometrical structures that describe the stream functions.

[1] $\nabla \mathbf{u} = 0$ for any compressible flow, where $\mathbf{u}$ is the velocity vector.

2.1 Characterization of Lagrangian coherent structures via finite Lyapunov exponents

Lagrangian finite time coherent structures (LCS)

On the definition of LCS: Consider the two-dimensional velocity field known for finite times

$$\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x}, t)$$

with $\mathbf{x} \in \mathbb{R}^2$, $t \in \mathbb{R}$, and $\mathbf{u}$ as a continuously differentiable function on $\mathbf{x}$ and t. The trajectory defined by this field starting at time t_0 in $\mathbf{x_0}$ will be referenced as $\mathbf{x}(t, t_0, \mathbf{x_0})$. The map $\mathbf{F^t_{t_0}} \colon \mathbf{x_0} \to \mathbf{x}(t_0 + t, t_0, \mathbf{x_0})$ relates initial positions at t_0 to later positions at time $t_0 + t$ in the phase space.

Dynamics generated by the velocity field can be explained in terms of material lines. Particles moved by the flow advection with time form curves that are called material lines. We will denote them by $\mathcal{L}(t)$.

Definition 2.1 (Material line)**.** One parameter family of class C^1 curves satisfying $\mathbf{F^t_{t_0}}(\mathcal{L}(t_0)) = \mathcal{L}(t)$.

Given a material line $\mathcal{L}(t) \in \mathbb{R}^2$, $(\mathcal{L}(t), t)$ is a two dimensional invariant manifold in the extended phase space $\mathbb{R}^2 \times \mathbb{R}$.

Attracting and repelling material lines must be distinguished in terms of local stabilities and instabilities. They are defined over an open time interval $\mathcal{I}$ within $\mathbb{R}$.

- **Attractive $\mathcal{L}$(t) over $\mathcal{I}$:** A material line such that any particle close to it exponentially converges to that manifold at some time in $\mathbb{R}$.

  Given a material line $\mathcal{L}(t)$ and a concrete trajectory $\mathbf{x}(t) \in (\mathcal{L}(t), t)$ consider $T_{\mathbf{x}}\mathcal{L}(t)$ as the two-dimensional tangent plane of $(x(t), t)$ in the extended phase space.

  $\mathcal{L}(t)$ is an attractive material line if there exists a constant $v > 0 \in \mathbb{R}$ and a smooth family of one-dimensional subspaces $E^s(t)$ in the extended phase space tangent to $T_{\mathbf{x}}\mathcal{L}(t)$ such that

  $$\nabla \mathbf{F}^t_{t_0}(\mathbf{x}(t_0))E^s(t_0) = E^s(t) \quad \text{and} \quad \|\nabla \mathbf{F^t_{t_0}}(\mathbf{x}(t_0))|_{E^s(t_0)}\| \leq e^{-v(t-t_0)}$$

  for $t_0, t \in \mathcal{I}$.

- **Repelling $\mathcal{L}$(t) over $\mathcal{I}$:** An attractive material line over $\mathcal{I}$ backwards in time. This means that any perturbation of the position of a particle initially on $\mathcal{L}(t)$ will exponentially diverge from $\mathcal{L}(t)$.

We call *finite time hyperbolic line* over $\mathcal{I}$ any material line that is attractive or repulsive.

The stability of the material lines changes over time. *Lagrangian coherent structure boundaries* are defined as material lines with locally the longest or shortest stability or instability times. Therefore, we can understand them as linearly stable or unstable material lines. In other words, *Lagrangian coherent structures* (LCS) retain the stability of the hyperbolic lines over time.

Given a material line L, we introduce the following scalar fields $T_{\mathbf{L}}(\mathbf{x_0}, t_0, t)$ to rigorously define LCS. For any initial condition $\mathbf{x_0}$ at time $t_0 \in [t_{-1}, t_1]$ we consider $T_{\mathbf{L1}}(\mathbf{x_0}, t_0, t_1) = \frac{1}{t_1 - t_0} \int_{\mathcal{I}_{\mathbf{L1}}} dt$, where $\mathcal{I}_{\mathbf{L1}}$ is the maximal open set within $[t_0, t_1]$ over which the trajectory $\mathbf{x}(t, t_0, \mathbf{x_0})$ remains in a material line of stability type **L1**.

Definition 2.2 (Lagrangian coherent structure)**.** Local extrem of the scalar fields $T_{\mathbf{L1,2}}(\mathbf{x_0}, t_0, t)$.

This definition yields to four distinct types of LCS. Let $\mathcal{C}(t)$ be a Lagrangian coherent structure boundary over time.

- $\mathcal{C}(t)$ as local maximizer of $\mathbf{T_L}$: Structure capturing or pushing away particles for locally the longest times on both sides of $\mathcal{L}(t)$.
- $\mathcal{C}(t)$ as local minimizer of $\mathbf{T_L}$: This case explains LCS's behaviour near a wall (physical barrier). Concretely their approach to the wall forward in time.

At larger distances from any wall, the coherent structures boundaries are local maximizers of $T_{\mathbf{L1}}$ and $T_{\mathbf{L2}}$ fields. In the framework of the ocean dynamics, potential walls are physical boundaries such as a coast or ice edge which are not considered in this project and therefore, LCS as local minimizers of time fields are no longer considered in our discussion.

On the detection of LCS: According to the previous definition, if we consider the maximal net growth of a unit vector transverse to a LCS, it has to be locally the largest over $\mathcal{I}$. This holds for either attractive or repelling structures.

Consider a repelling structure $\mathcal{C}(t)$ and $\mathbf{x_0}$ a point in $\mathcal{C}(t_0)$. $\nabla_{\mathbf{x_0}}\mathbf{F^t_{t_0}}$ is the linearized flow map around $\mathbf{x_0}$ so the following equation describes the propagation of a unit vector $\mathbf{e_{t_0}}$ selected at $\mathbf{x_0}$ not tangent to $\mathcal{C}(t_0)$ along the trajectory $\mathbf{x}(t, t_0, \mathbf{x_0})$. ($\mathbf{e_{t_0}}$ is identified with $\mathbf{v}$.)

$$\mathbf{e}_t(\mathbf{x_0}) = (\nabla_{\mathbf{x_0}}\mathbf{F^t_{t_0}})\mathbf{e_{t_0}}.$$

We concluded that repelling coherent structures are the maximizers of $|\mathbf{e_{t_0}}(\mathbf{x_0})|$ over all possible $\mathbf{e_{t_0}}$ and $\mathbf{x_0}$. We define $\mathcal{E}^t_{t_0}(\mathbf{x_0})$ as the maxim resulting propagation over all choices of $\mathbf{e_{t_0}}$.

$$\mathcal{E}^t_{t_0}(\mathbf{x_0}) = \max_{|\mathbf{e}|=1} |(\nabla_{\mathbf{x_0}}\mathbf{F^t_{t_0}})\mathbf{e_{t_0}}| \equiv \|(\nabla_{\mathbf{x_0}}\mathbf{F^t_{t_0}})\|,$$

where $\| \ \|$ is the operator norm $\|\mathbf{A}\| = \max_{|\mathbf{x}|=1} |\mathbf{Ax}|$ for a general matrix $\mathbf{A} \in \mathbb{R}^2$. By definition, $\|\mathbf{A}\|$ is the square root of the maximal eigenvalue of the positive matrix $\mathbf{A}^T\mathbf{A}$ namely $\|\mathbf{A}\| = \lambda_{\max}(\mathbf{A}^T\mathbf{A})$, so $\mathcal{E}^t_{t_0}(\mathbf{x_0})$ can be rewritten as

$$\begin{aligned}\mathcal{E}^t_{t_0}(\mathbf{x_0}) &= \sqrt{\lambda_{\max}((\nabla_{\mathbf{x_0}}\mathbf{F^t_{t_0}})^T(\nabla_{\mathbf{x_0}}\mathbf{F^t_{t_0}}))},\\ \dot{\mathbf{x}}(t) &= \Lambda\mathbf{x}(t),\\ \mathbf{x}(t) &= \mathbf{x}(t_0)e^{\Lambda(t-t_0)} = \mathbf{x}(t_0)\mathcal{E}^t_{t_0}(\mathbf{x_0}).\end{aligned}$$

This is a measure of the maximal growth over the most repelling direction in $\mathbf{x_0}$. $\mathbf{x_0}$'s with large values of $\mathcal{E}^t_{t_0}(\mathbf{x_0})$ define the LCS.

Finite size and time Lyapunov exponents approach to LCS

We define the finite time Lyapunov exponent (FTLE) at $(t, t_0, \mathbf{x_0})$ as $\Lambda(t, t_0, \mathbf{x_0})$ following the next equation

$$\Lambda(t, t_0, \mathbf{x_0}) = \frac{1}{2(t-t_0)} \log_e\Big(\lambda_{\max}((\nabla_{\mathbf{x_0}}\mathbf{F^t_{t_0}})^T(\nabla_{\mathbf{x_0}}\mathbf{F^t_{t_0}}))\Big).$$

Following the formality in the previous section, the local maxima of $\mathcal{E}_{t_0}^t(\mathbf{x_0})$ coincides with the local maxima of the finite time Lyapunov exponent since they are related as

$$\mathcal{E}_{t_0}^t(\mathbf{x_0}) = e^{\Lambda(t,t_0,\mathbf{x_0})(t-t_0)}.$$

Therefore, $\mathbf{x_0'}s$ locally maximizing of FTLE is a good approach to LCS. As the relative stretching tends to grow rapidly, it is more convenient to work with FTLE that are not attenuated by the factor $(t - t_0)$ than to directly work with $\mathcal{E}_{t_0}^t(\mathbf{x_0})$.

Finally, we introduce the concept of finite size Lyapunov exponents (FSLE) as an equivalent to FTLE. Accordingly to [7] they are defined as

$$\Pi(r, t_0, \mathbf{x_0}) = \frac{\log_e(r)}{2(t - t_0)}, \tag{1}$$

where $t - t_0$ is the minimum time for which $r = \lambda_{\max}((\nabla_{\mathbf{x_0}}\mathbf{F_{t_0}^t})^T(\nabla_{\mathbf{x_0}}\mathbf{F_{t_0}^t}))$.

It is key to understand the intuitive idea of finite Lyapunov exponents as a measure of the exponential growth of small perturbations. In this sense, a more intuitive definition equivalent to (1) is associating the FSLE to the maximum value of

$$\boldsymbol{\lambda}(\mathbf{x_0}, t_0, \boldsymbol{\delta_0}, \delta_f) = \frac{1}{\tau}\log\left(\frac{\delta_f}{|\boldsymbol{\delta_0}|}\right)\frac{\boldsymbol{\delta_0}}{|\boldsymbol{\delta_0}|}$$

over all the accessible directions of $\boldsymbol{\delta_0}$.

τ is the backward time that the two trajectories take starting at time t_0 at $\mathbf{x_0}$ and $\mathbf{x_0}+\boldsymbol{\delta_0}$ to respectively reach the prescribed separation $\delta_f > |\boldsymbol{\delta_0}|$.

FTLE and FSLE have units of time^{-1}. Both exponents represent a finite time, respective size, description. Therefore only under certain conditions do FSLE ridges imply close FTLE ridges which in turn indicate the existence of an hyperbolic LCS [4]. In this work we assume their total equivalence as detectors of LCS so FSLE highlight the transport barriers that control the horizontal exchange of water in and out of eddy cores.

For ocean signals, we only dispose of finite and discrete time sequences of images of discrete spatial resolution, namely Δx. This determines the implementation of algorithms to compute FTLE and FSLE. According to the previous definitions, FTLE are defined over a prescribed time while FSLE are defined over a prescribed final separation. Concretely, FSLE depend on the relative size of $\boldsymbol{\delta_0}$ and δ_f against Δx. Choosing $\boldsymbol{\delta_0} \ll \Delta x$ would imply that no point lying further than δ_0 of any grid point would ever be tested. Therefore, in the regions where the signal is stretching, the method would give a discontinuous sampling of the structures. Instead, if $\boldsymbol{\delta_0} \gg \Delta x$, the algorithm would lead to a loss of spatial resolution since the same smeared stretching manifold would be detected on several grid points. The previous simple argument shows the convenience of using a value of $\boldsymbol{\delta_0}$ close to Δx. The second length scale δ_f determines the size of the structures willing to be evaluated. This is the reason why we choose FSLE and not FTLE. We prioritize controlling the length of the structures we want to detect above their duration.

2.2 Singularity analysis for remote sensed images of the ocean

Singularity analysis is the process of obtaining a dimensionless measure of the degree of irregularity at each point of a given signal $s(\mathbf{x})$. This measure is known as the singularity exponent and it refers to the analysis of singularities[2] of differentiable functions.

[2]Singularities as points, where the mathematical object ceases to be well-behaved by lacking differentiability or analytically.

Singularity exponents

Different implementations of computation for singularity exponents are possible. Here we expand on the most usual meaning in the theory of complex systems. This is understanding SE as the continuous extension of continuity or differentiability. SE are dimensionless and track transitions independently of their amplitude. Therefore, even subtle structures can be detected on their images.

Consider the pseudo-Taylor expansion ([9, 11]) of the signal

$$s(\mathbf{x}+\mathbf{r}) - s(\mathbf{x}) = \alpha(\mathbf{x}) r^{H(\mathbf{x})} + \mathcal{O}(r^{H(\mathbf{x})}), \quad r \to 0.$$

If any point $\mathbf{x}$ in the signal behaves according to the previous expansion for a concrete value of $\alpha(\mathbf{x})$, the non necessarily integer exponent $H(\vec{x})$ is defined as the singularity exponent. It is also known as the Hölder exponent measuring the degree of regularity (positive value) or irregularity (negative value) of the signal. Negative exponents characterize regularity. Exponents vanishing to zero imply continuity but non-differentiability. Values in the range $(0,1)$ are found in points where the signal is more regular than a continuous one but still not differentiable. Finally, 1, 2, and further k integers imply k-differentiability.

Any continuous function $\psi(\mathbf{x})$ can be recognised as a wavelet or mother wavelet. For efficiency reasons, common wavelets are characterized by properties such as differentiability, orthogonality, compact support, symmetry, and vanishing moment [5].

Definition 2.3. The *wavelet projection* or *continuously wavelet transform* of a signal $f(\mathbf{x})$ on the mother wavelet ψ in $\mathbf{x}$ and with a scale scope r is

$$\begin{aligned} \mathcal{T}_\psi s(\mathbf{x}, r) &= \int ds(\mathbf{y}) \frac{1}{r^d} \psi\left(\frac{\mathbf{x}-\mathbf{y}}{r}\right), \\ \mathcal{T}_\psi s(\mathbf{x}, r) &= \alpha_\psi(\mathbf{x}) r^{H(\mathbf{x})} + \mathcal{O}(r^{H(\mathbf{x})}), \quad r \to 0, \\ \mathcal{T}_\psi \nabla s(\mathbf{x}, r) &= \alpha_\psi(\mathbf{x}) r^{h(\mathbf{x})} + \mathcal{O}(r^{h(\mathbf{x})}), \quad r \to 0, \end{aligned} \tag{2}$$

where d is the dimension of the signal domain.[3] For $\psi(\mathbf{x})$ to be a wavelet for the continuous wavelet transform, there is an admissibility criterion implying that the first order moment must vanish to zero.

The wavelet projection is the convolution of the signal with a re-sized version of the mother wavelet $\psi(\mathbf{x})$. The resolution parameter r regulates the range of the wavelet. For its construction, it conveys redundant information but is well adapted to detect transitions in data.

It is proved that for signals obeying (2), their wavelet projections also scale with the same power law over r (see [3]). This power law is well adapted to filter long range correlations and just detects local structure that represent the continuous scale of changes.

In practice, the computation of the exponents from real discrete data has a rough resolution even using wavelet transformations. Wavelet projections of the modulus of the gradient lead to a more precise determination of SE with improved spatial resolution. For any signal scaling as a power law in r, its gradient scales similar. Understanding the singularity exponents as a continuous measure of differentiability, the gradient operator lessens by one unit the differentiability degree. This implies that the SE of the gradient, let it be $h(\mathbf{x})$, is related to the SE of the proper signal $H(\mathbf{x})$ like

$$h(\mathbf{x}) = H(\mathbf{x}) - 1.$$

[3] In the two dimensional case we work, $d = 2$.

In this project we will consider SE associated to gradient measures. Therefore, the nomenclature for SE will be $h(\mathbf{x})$. There exists a theoretical bound for $h(\mathbf{x})$ in range $(-1, 2)$.[4]

Multifractal signal and the most singular component (MSC)

One way to show that a given image possess multifractal structure is to construct a positive measure μ which assigns positive value to any set $\mathcal{A}$. The measure must take into account any sharp transitions in $\mathcal{A}$. We define the measure in terms of its density like

$$\mu(\mathcal{A}) = \int_{\mathcal{A}} d\mu(\mathbf{x}) \quad \text{with} \quad d\mu(\mathbf{x}) \equiv |\nabla s|(\mathbf{x})\, d\mathbf{x}.$$

The behaviour of any particular point $\mathbf{x}$ can be characterized in terms of the evolution of the measure of balls centered on $\mathbf{x}$ of radius r denoted as $\mathcal{B}_r(\mathbf{x})$.

The measure μ defines a *multifractal* if it is characterized by unique exponent $h(\mathbf{x})$ and coefficient $\alpha(\mathbf{x})$ such that

$$\mu(\mathcal{B}_r(\mathbf{x})) = \alpha(\mathbf{x}) r^{d+h(\mathbf{x})} + \mathcal{O}(r^{d+h(\mathbf{x})}). \tag{3}$$

$\mathcal{O}(r^{d+h(\mathbf{x})})$ is negligible in comparison with $r^{d+h(\mathbf{x})}$.

The exponent $h(\mathbf{x})$ in (3) attains the definition of singularity exponent for the gradient measure μ.

$\alpha(\mathbf{x})$ depends on the particular metrics in the definition of the ball and the scaling unit for r, but does not provide information about changes in scale. All the information about the evolution under changes in r is contained in the SE which is independent of metrics and scaling units.

As discussed in the previous section, applying a wavelet projection over the measure leads to the same power law that permits a more efficient estimation of the exponents. This is

$$\mathcal{T}_\psi \mu(\mathbf{x}, r) = \alpha_\psi(\mathbf{x}) r^{h(\mathbf{x})} + \mathcal{O}(r^{h(\mathbf{x})}) \quad \text{equivalent to} \quad m(\mathbf{x}, r) = \alpha(\mathbf{x}) r^{h(\mathbf{x})} + \mathcal{O}(r^{h(\mathbf{x})}) \tag{4}$$

for any $m(\mathbf{x}, r)$ defined measure of unpredictability.

Definition 2.4. A *fractal component* F_h is defined as the set of points of an image having the same exponent h.

$$F_h = \{\mathbf{x} \text{ such that } h(\mathbf{x}) = h\}.$$

The decomposition of a signal as the union of different fractal components is called a multifractal decomposition of the signal.

Fractal components are of a very irregular nature. The odd arrangement of the points in fractal sets can be characterized by counting the number of points contained inside a given ball of radius r. We define this number as $N_r(h, \Delta_h)$, where Δ_h is a threshold for admissibility in F_h. As $r \to 0$ the following power law holds ([10])

$$N_r(h, \Delta_h) \approx r^{D(h)}.$$

The exponent $D(h)$ quantifies the size of the set of pixels with singularity h as the image is covered with small balls of radius r. It is known as the *fractal dimension* of the associated fractal component F_h. The

[4] The lower bound comes from the physical limit that no point from a signal tracing of the ocean dynamics can infinitely diverge. The upper bound is consequence of the lower bound and the transitional invariance of the signal.

function $D(h)$ defined $\forall h \in (-1, 2)$ for signals of ocean tracers is called the dimension spectrum of the multifractal image. Fractal components usually range from dimension one, $D(h) \approx 1$, for the most singular values (curve-like components), to dimension $D(h) \approx 2$ for higher exponents that extend on less definite areas.

The multifractal behaviour allows a strong hierarchical organization in images. This organization is explained in terms of the most singular components (MSC).

Definition 2.5. The *most singular component* (MSC)[5] is the fractal component

$$F_\infty = \{\mathbf{x} \text{ such that } h(\mathbf{x}) \in \,]h_\infty - \Delta, h_\infty + \Delta[\},$$

where h_∞ is the minimal value over the domain and Δ a threshold.

There exists a unique operator associated to the most singular component (see [8]) that reconstructs the image. For that, the MSC is recognised as the more informative component.

Estimation of singularity exponents using the unpredictable point manifold

Signal reconstruction from a partial set of the gradient is the key concept to define a wavelet that precisely identifies SE. We discussed how one can properly reconstruct any signal from the gradient of MSC points. Therefore MSC can be identified as the less predictable set of points. The unpredictable point manifold (UPM) consists in the set of points that cannot be reconstructed from others so it can be associated to the MSC.

To numerically characterize the unpredictable point manifold, a measure of unpredictability for each point $\mathbf{x}$ has to be settled. It can be identified as $\mathcal{T}_\psi \mu$ in (4) and singularity exponents can be derived from it.

The following is a punctual estimation of the singularity exponent $h(\mathbf{x})$ defined in (4).

$$h(\mathbf{x}) = \frac{\log\left(\frac{\mathcal{T}_\psi \mu(\mathbf{x}, r_0)}{\langle \mathcal{T}_\psi \mu(\mathbf{x}, r_0)\rangle}\right)}{\log r_0} + \mathcal{O}\left(\frac{1}{\log r_0}\right), \quad h(\mathbf{x}) \approx \frac{\log\left(\frac{m(\mathbf{x})}{\langle m \rangle}\right)}{\log r_0},$$

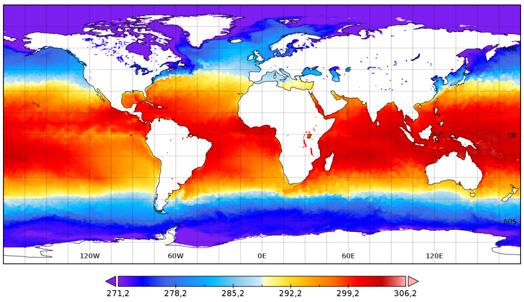
where $m(x)$ is an effective UPM-measure for 2D images. For more information on the algorithm, please contact the author. The algorithm is part of an internal note of the Institute of Marine Sciences in Barcelona.

3. Lyapunov and singularity exponents in a global scale and empirical functional relation between them

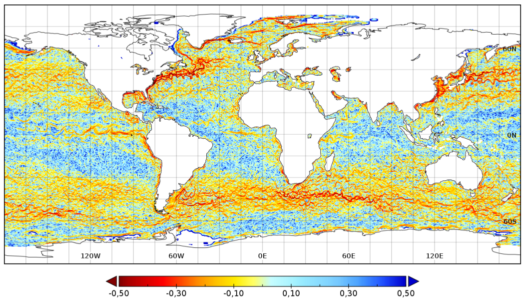
The results presented here are obtained analysing the following data: (i) *Backward-in-time, finite size Lyapunov exponents and orientations of associated eigenvector*. The exponents are distributed on a 0.25^0 grid on a global coverage and have units of day^{-1}. Figure 2a contains a map of FSLE from this product.

[5] MSC might also be found as the most singular manifold (MSM) in the bibliography.

(ii) SST and ADT global, two-dimensional remote sensed signals from Copernicus Marine Service with spatial resolution of 0.25^0 and daily temporal resolution. The computation of the exponents is based on an estimation of the unpredictable measure (image processing) introduced in 2.2.



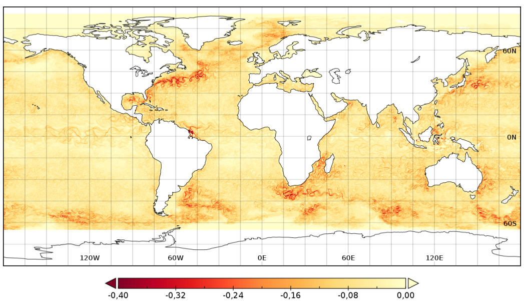
(a)



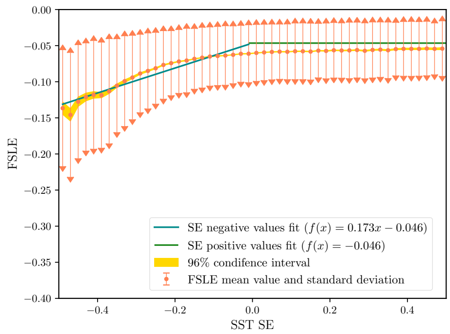
(b)

Figura 1: (a) SST global image from [2] *Global Ocean OSTIA Sea Surface Temperature and Sea Ice Analysis Results* for January 25th, 2022. Temperatures in the color bar are expressed in Celsius degrees. (b) Global image of the singularity exponents (SE SST) computed from (a).

The SST SE map is qualitatively as expected. There are some areas of the ocean known to be the most energetic ones and having the most important streams. Some of these areas are the East coast of North America, with the Gulf Stream, and the Antarctic Circumpolar Current that connects the Southern Ocean. We see in our results that these areas have the most negative exponents with well defined fractal components. In the areas showing the most negative exponents, complex undulated shapes, wave-like instabilities, eddy-like patterns, and intense vortices are distinguished. Those areas are associated to the most singular component. Their organization in curve-like structures justifies their fractal dimension to be $D(h) \approx 1$. Areas with positive exponents are less defined and do not present a clear structure nor intense vortices. The organization in fractal components is more confusing so their fractal dimension is closer to $D(h) \approx 2$; the maximal possible.



(a)



(b)

Figura 2: (a) Backward-in-time, finite size Lyapunov exponents and orientations of associated eigenvectors product in [1] for January 25th, 2022. (b) Global fit and piecewise function explaining the relationship between FSLE and SST SE of the global maps 2a and 1b.

Analysing the FSLE, one can distinguish boundaries of coherent structures that coincide with the previously described as the most energetic areas. The boundaries are defined by curve-like patterns with the most negative values of the exponents (i.e. the regions with the strongest exponential attractive material lines). Visually, the structures detected by singular analysis and Lagrangian coherent structures seem to effectively have a correspondence, at least with their location on the maps. The analysis of data by density plots shows that most of the SE exponents are found in the range $[-0.1, 0.1]$ and the Lyapunov ones in the range $[-0.1, -0,05]$. In fact, the most negative values of FSLE are only associated to the most negative values of SST SE. Nevertheless, negative values of SE SST are present in the whole range of FSLE values.

To define an empirical functional relation between SST SE and FSLE we compute the density and normalized density plots. Then we take the average over FLSE values (see Figure 2b). This is considering the mean value of the FSLE values x_i associated to each SE (SE values ordinated in bins of 0.02 units width). Each point includes the standard deviation representation of the average value it represents $\sigma = \sqrt{\frac{\sum_{n=1}^{N}(x_i - \bar{x})^2}{N-1}}$ and the 96% interval of confidence. Then, the functional relation (FSLE in terms of SE) is assessed over this averaged plot, constructing a continuous piecewise function. For the negative domain of SE we have linearly fitted the values from the previous average. We have associated the constant value $f(SE = 0)$ to the positive SE. f represents the linear fit. This is only a preliminary estimation of the relation.

The coefficient of the regression between FSLE and negative SE SST is $r^2 = 0.88$. We observe that the relationship between these two variables is not completely linear. However, in this analysis we simplify and we keep on using this preliminary linear fit. This linear fit has some limitations: (i) the average is a measure of the most frequent value for a given SST SE value, but the standard deviation shows that there is a large variability that will not be taken into account in this approach. (ii) the mean value is affected by outliers and in this case produces an overestimation of the most negative FSLE values. Besides the limitations of the linear fit approach, the data itself also has some accuracy limitations and the methods applied for deriving FSLE and SE also add uncertainty to the full process.

Reconstruction of FSLE from SST SE

We have reconstructed the FSLE's map using the SST SE map and the functional relation established in Figure 2b. We observe several limitations of the applied approach: (i) By definition, the areas having the less singular exponents, namely positive, are associated to constant, close to zero values of FSLE. This is a first limitation of our reconstruction because the real FSLE map, although having areas with light variability does not have any constant part. (ii) The range of variation of the reconstructed FSLE $(-0.2 : 0)$ is lower than the one of the original FSLE values $(-0.4 : 0)$. This is in part because of the effect of the linear fitting that does not allow recovering the most negative values. In addition, the use of the average FSLE to define the functional relation leads to a limited accuracy given by the variability of the FSLE around this average. (iii) Since the functional relation we establish is linear, the negative SE in the reconstructed map has the same pattern as the original SE map 1b. Here we compare the FSLE original map 2a and the reconstructed one. The figure shows the percentage of points where the difference between reconstructed and actual FSLE values (y-axis) is lower than a threshold (x-axis) for the same map used in the linear fit (no independent assessment, orange line) and for a different map (independent assessment, blue line). Both lines remain quite close one to the other with a small degradation for the independent map. We see that with this approach the reconstructed FSLE describes the 70% of the map with an accuracy of 0.05.

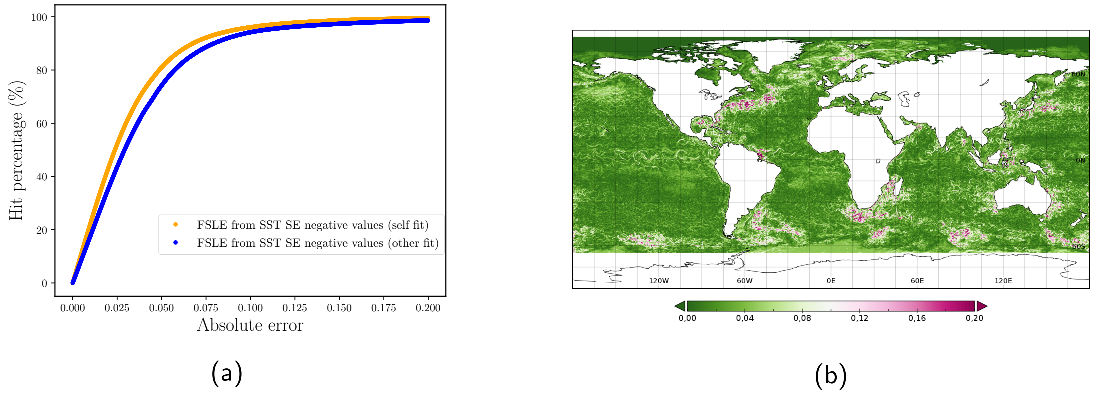


Figura 3: (a) Hit percentage for the global reconstructed map for the same day the functional relation has been derived the 25th January 2022 (self fit), and SE SST map of another day (8th June, 2021). (b) Absolute error between the FSLE reconstructed map and the AVISO+ map 2a.

The map of the differences between reconstructed and actual FSLE values reveals that the reconstruction especially fails to predict the most negative exponents (for the limitations above described). See Figure 3b.

3.1 High energic regions

Finally, we assess up to which extent the global relation found in Figure 2b is also accurate at regional scales. We have analysed the following three areas: (i) Gulf Stream area in East Coast of North America: $[75W, 30W]$ $[30N, 60N]$. (ii) Angulas current area in South Africa: $[0E, 45E$ E] $[60S, 30S]$. (iii) South Atlantic current area in South America: $[75W, 30W]$ $[60S, 30S]$.

The patterns of the functional relation are the same. The reduced number of pixels used in those regional plots make them more rough than the global ones. In future work we plan to use a larger time series of maps (for both global and regional assessment) in order to derive more robust density plots.

The linear fitting of the average FSLE value as function of the SST SE is also consistent with the one of the global maps. However we observe a constant offset between the regional and the global fitting. We associate these differences to the background high energy level of the selected regions.

Comparison between singularity exponents from ADT and SST

Finally, we have repeated the previous analysis but changing the SST SE for the ADT SE. From a theoretical point of view, both SE should be equal and both of them should be in correspondence with the streamlines. However, we observe that the accuracy of the different acquisitions is also a source of uncertainty affecting our results.

Qualitatively, comparing the maps representing the exponents they present the same large scale features. However, some important differences between both exponents appear. The most negative SST SE are organized in smoother and narrower curves with less negative SE values than the ADT SE. This is because of the effective spatial resolution of each product that is defined by the accuracy of the different instrument measuring the data and the different data processing algorithms.

4. Conclusions

The first objective of this paper is to understand two different mathematical concepts that describe the state of the ocean.

The second objective is to assess up to which extent the structures defined by the LCS and by the different multifractal components are related. The most negative FLSE are associated with the transport barriers. In our comparison we observe that positive SST SE are associated with the FLSE with values closest to zero, while the most negative SST SE present a monotonic growing relation with FSLE. We have linearly fitted negative SST SE with FSLE and we have assessed this relation. This approach allows representing 70% of the FSLE with an accuracy of 0.05. This analysis is the first step to understanding how these two exponents are related. During the study some limitations have been identified: The linear fitting overestimates the most negative FSLE. In the future we plan to fit with nonlinear functions to better estimate the most negative FSLE. We observe constant offset between the regional and the global fitting. We associate these differences to the background high energy level of the selected regions. This suggests that depending on the accuracy required in future applications, having regional fittings could be more appropriated than global ones.

We have selected the SST as the main ocean variable to be used to compute the reconstructed FS-LE. This intrinsically assumes that the SE from any ocean variable are equal (this assumption is based on [6]). However, in practice, because of the different accuracy of the available data, we have also observed that there are significant differences between the SST SE and ADT SE. This leads to distinct functional relations between the FSLE and the SE. In order to address this in the future we plan to use an extended temporal series of satellite images. This would allow us to derive a more robust relationship between FSLE and SE. With this we will assess which are the strengths and limitations of using one ocean variable or the other. Besides, having larger temporal series will allow us to assess the temporal stability of the relationship between FSLE and SE. Finally, we have also provided an estimation of the uncertainty associated to the reconstructed FSLE which is based on the difference between actual and reconstructed values. There are other metrics that could provide valuable information for marine and maritime applications. For example the correspondence among the geometrical structures of both Lyapunov and singular exponents.

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Outerplanar partial cubes

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Resum *(CAT)*
Els partial cube-menors són una analogia de la noció de menors als partial cubes. En aquest article determinem el conjunt de pc-menors de les classes dels partial cubes outerplanars i els partial cubes sèrie-paral·lel. Aquest és el primer resultat d'aquest tipus per als partial cubes d'una classe tancada per menors.

Abstract *(ENG)*
Partial cube-minors are an analogue of graph minors in partial cubes. We determine the set of forbidden partial cube minors of the classes of outerplanar and series-parallel partial cubes. This is the first result of this type for the partial cubes in a minor closed graph class.

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1. Introduction

Denote by Q_d the hypercube graph of dimension d, i.e., its vertices are the elements of $\{0,1\}^d$ and two vertices are adjacent if they differ in exactly one entry. Partial cubes are the graphs that admit an isometric embedding into a hypercube; see Figure 1 for examples. They were introduced by Graham and Pollak [19] in the study of interconnection networks, form an important graph class in media theory [18], frequently appear in chemical graph theory [17, 20], and quoting [21], *present one of the central and most studied classes in Metric Graph Theory*. Some classes of partial cubes that are studied within Metric Graph Theory include median graphs [4], bipartite cellular graphs [3], hypercellular graphs [12], Pasch graphs [11], netlike partial cubes [24], and two-dimensional partial cubes [13]. Partial cubes arise also from geometry as graphs of regions of hyperplane arrangements in $\mathbb{R}^d$ [6], tope graphs of oriented matroids (OMs) [7], 1-skeleta of CAT(0) cube complexes [4], and more generally: tope graphs of complexes of oriented matroids [5].

An interesting structural feature of partial cubes is that they admit a natural minor-relation (pc-minors for short) consisting of restrictions and contractions, which are special forms of deletion and contraction in the graph. Many important classes of partial cubes are closed under taking pc-minors. Analogously to graph minors, given a pc-minor closed class there exists a list of excluded pc-minors of the class. Contrary to the situation of graph minors [25] for pc-minors this list might be infinite. If the list is finite, this also allows for a polynomial time recognition algorithm of the class [23]. Even if the list is infinite, determining it can yield insight into the class. All excluded minors are known for tope graphs of complexes of oriented matroids [23], two-dimensional partial cubes [13], median graphs, bipartite cellular graphs, hypercellular graphs, and Pasch graphs [12]. See [22, Chap. 7.5] for more related material. Since pc-minors are special graph minors, one source for pc-minor closed classes of partial cubes is the class of partial cubes in a minor-closed graph class. In the present paper we analyze the first non-trivial instance of such a class: partial cubes that are outerplanar partial cubes, i.e., they admit a crossing-free drawing in the plane such that all vertices lie on the outer face. We give a full description of its infinite list of excluded pc-minors (Theorem 4.21). Further, we obtain the list for series-parallel partial cubes (Theorem 4.22). Our proof uses the excluded minors for these classes [9] and we discuss in Section 5 possible extensions to other pc-minor closed classes. This short version omits some proofs, which can be found in [26].

2. Partial cubes

All graphs $G=(V,E)$ occurring in this paper are simple, connected, and finite. The *distance* $d(u,v) := d_G(u,v)$ between two vertices u and v is the length of a shortest (u,v)-path, and the *interval* $I(u,v)$ between u and v consists of all vertices on shortest (u,v)-paths: $I(u,v) := \{x \in V : d(u,x)+d(x,v) = d(u,v)\}$. If this causes no confusion, we will denote the distance function of G by d and not d_G. An induced subgraph of G is called *convex* if it includes the interval of G between any two of its vertices. An induced subgraph H of G is *isometric* if the distance between any pair of vertices in H is the same as that in G. In particular, convex subgraphs are isometric. A graph $G=(V,E)$ is *isometrically embeddable* into a graph $H=(W,F)$ if there exists a mapping $\varphi \to V \to W$ such that $d_H(\varphi(u),\varphi(v)) = d_G(u,v)$ for all vertices $u,v \in V$, i.e., $\varphi(G)$ is an isometric subgraph of H. A graph G is called a *partial cube* if it admits an isometric embedding into the hypercube Q_d. From now on, we will suppose that a partial cube $G=(V,E)$ is an isometric subgraph of the hypercube Q_d, i.e., we will identify G with its image under the isometric embedding and its vertices will often be denoted as elements of $\{0,1\}^d$. The minimal d such that G embeds isometrically into Q_d is called the *(isometric) dimension* of G. The edges of G are partitioned into so-called Θ-*classes*, i.e., $e\Theta e'$ iff both edges correspond to a switch in the same coordinate of Q_d. Denote by $\mathcal{E} = \{E_i : i \in [d]\}$ the equivalence classes of Θ. Sometimes we will refer to Θ as a function $\Theta\colon E(G) \to \mathcal{E}$. The Θ-classes can be characterized intrinsically and do not depend on the embedding [16].

2.1 Partial cube minors

Let $G = (V, E)$ be an isometric subgraph of the hypercube Q_d. Given $f \in [d]$, an *elementary restriction* consists in taking one of the two connected components $\rho_{f^-}(G)$ and $\rho_{f^+}(G)$ of $G \setminus E_f$. These graphs are isometric subgraphs of the hypercube $Q([d] \setminus \{f\})$. Now applying twice the elementary restriction to two different coordinates f, g, independently of the order of f and g, we will obtain one of the four (possibly empty) subgraphs. Since the intersection of convex subsets is convex, each of these four subgraphs is convex in G and consequently induces an isometric subgraph of the hypercube $Q([d] \setminus \{f, g\})$. More generally, a *restriction* is a convex subgraph $\rho_A(G)$ of G, where $A \in \{+, -, 0\}^{[d]}$, obtained by iteratively applying $\rho_{e^{A_e}}$ for all $A_e \neq 0$. The following is well-known:

Lemma 2.1 ([1, 2])**.** *The set of restrictions of a partial cube G coincides with its set of convex subgraphs. In particular, the class of partial cubes is closed under taking restrictions.*

For $f \in [d]$, we say that the graph G/E_f obtained from G by contracting the edges of the equivalence class E_f is an (f-)*contraction* of G. For a vertex v of G, we will denote by $\pi_f(v)$ the image of v under the f-contraction in G/E_f, i.e., if uv is an edge of E_f, then $\pi_f(u) = \pi_f(v)$, otherwise $\pi_f(u) \neq \pi_f(v)$. We will apply π_f to subsets $S \subset V$, by setting $\pi_f(S) := \{\pi_f(v) : v \in S\}$. In particular, we denote the *f-contraction* of G by $\pi_f(G)$. It is well-known and follows from the proof of the first part of [10, Thm. 3] that $\pi_f(G)$ is an isometric subgraph of $Q([d] \setminus \{f\})$. Since edge contractions in graphs commute, i.e., the resulting graph does not depend on the order in which a set of edges is contracted, we have:

Lemma 2.2. *Contractions commute in partial cubes, i.e., if $f, g \in [d]$ and $f \neq g$, then $\pi_g(\pi_f(G)) = \pi_f(\pi_g(G))$. Moreover, the class of partial cubes is closed under contractions.*

Consequently, for a set $A \subset [d]$, we can denote by $\pi_A(G)$ the isometric subgraph of $Q([d] \setminus A)$ obtained from G by contracting the classes $A \subset [d]$ in G. Finally, we have:

Lemma 2.3 ([12])**.** *Contractions and restrictions commute in partial cubes, i.e., if $f, g \in [d]$ and $f \neq g$, then $\rho_{g^+}(\pi_f(G)) = \pi_f(\rho_{g^+}(G))$.*

The previous lemmas show that any set of restrictions and any set of contractions of a partial cube G provide the same result, independently of the order in which we perform the restrictions and contractions. The resulting graph G' is also a partial cube, and G' is called a *partial cube-minor* (or *pc-minor*) of G.

2.2 Expansions and Cartesian products

A partial cube G is an *expansion* of a partial cube G' if $G' = \pi_f(G)$ for some equivalence class f of $\mathcal{E}(G)$. More generally, let G' be a graph containing two isometric subgraphs G'_1 and G'_2 such that $G' = G'_1 \cup G'_2$, there are no edges from $G'_1 \setminus G'_2$ to $G'_2 \setminus G'_1$, and $G'_0 := G'_1 \cap G'_2$ is nonempty. A graph G is an *isometric expansion* of G' with respect to G'_0 if G is obtained from G' by replacing each vertex v of G'_1 by a vertex v_1 and each vertex v of G'_2 by a vertex v_2 such that u_i and v_i, $i = 1, 2$, are adjacent in G if and only if u

and v are adjacent vertices of G'_i and v_1v_2 is an edge of G if and only if v is a vertex of G'_0. Every partial cube can be obtained from a single vertex by a sequence of expansions [10].

The *Cartesian product* $F_1 \square F_2$ of two graphs $F_1 = (V_1, E_1)$ and $F_2 = (V_2, E_2)$ is the graph defined on $V_1 \times V_2$ with an edge $(u, u')(v, v')$ if and only if $u = v$ and $u'v' \in E_2$ or $u' = v'$ and $uv \in E_1$. Cartesian products of partial cubes are partial cubes. It follows immediately from the definitions that:

Lemma 2.4. *A partial cube G is an expansion of the partial cube G' if and only if $G' \subseteq G \subseteq G' \square K_2$ are isometric subgraphs.*

3. The excluded minors

A graph is *outerplanar* if it admits a planar drawing for which all vertices lie on the outer face of the drawing. This class is minor-closed hence also outerplanar partial cubes have a set of excluded pc-minors, which we will denote by Ω. Denote by $L := K_{1,3} \square K_2$ the *book graph* and by $n \geq 3$, G_n is the *gear graph*, i.e., the graph formed by $2n + 1$ vertices: an even exterior cycle of length $2n$ and a center vertex adjacent to one bipartition class of the cycle.

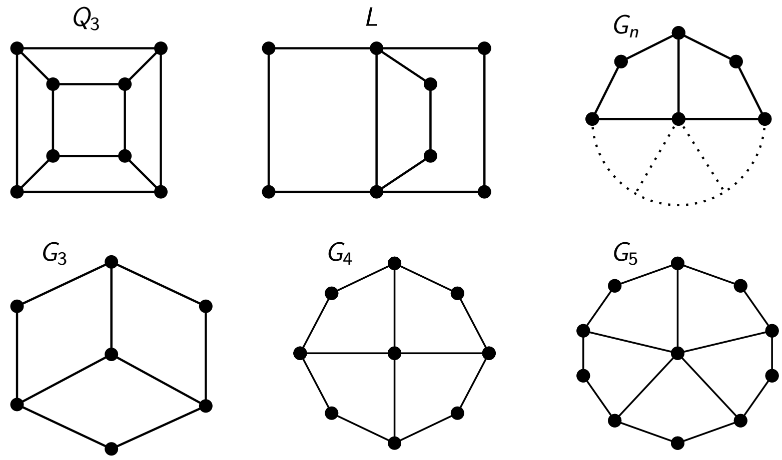

Figure 1: The cube, the book graph, and the infinite family of gear graphs.

It is easy to see that all the partial cubes in Figure 1 are pc-minor minimal non-outerplanar. Our main result is that they are the only such graphs. The proof will occupy the rest of this paper.

4. Main proof

4.1 Preparation

Before we get into the proof, we need some lemmas whose proofs are omitted in this short version.

Lemma 4.1. *If $G \in \Omega$, then G is planar.*

Let G be a graph, let F be a set of edges, let $H \subseteq G$ be a subdivision of a certain graph K. We say that F *destroys* H if H/F is not a subdivision of K. We say that F *destroys* K if G/F does not contain any subdivision of K as a subgraph.

Lemma 4.2. *Let $G \in \Omega$, let E_i be a Θ-class. Then E_i destroys K_4 or $K_{2,3}$. In particular, if $H \subseteq G$ is a subdivision of K_4 or $K_{2,3}$, then E_i destroys H.*

If $H \subseteq G$ is a subgraph, we refer to the induced subgraph by $V(H)$ as the induced subgraph by H and denote it as $G[H]$.

Lemma 4.3. *Let G be a graph. Let $H \subseteq G$ be a subdivision of a certain graph K. Let F be a matching. Then $F \backslash E(G[H])$ does not destroy H.*

If $H \subseteq G$ is a subgraph and F is a set of edges of G, then we denote $F[H] := F \cap E(G[H])$.

Lemma 4.4. *Let $G \in \Omega$. Let $H \subseteq G$ be a subdivision of K_4 or $K_{2,3}$. Let E_i be a Θ-class. Then $E_i[H] \neq \emptyset$.*

4.2 Three lemmas

Lemma 4.5. *Let G be a partial cube containing a subdivision of $K_{2,3}$ or K_4 such that no pc-minor of G does. If $\dim(G) \leq 3$, then $G = G_3$ or $G = Q_3$.*

Proof. Partial cubes of dimension 0, 1, and 2 are all outerplanar. For dimension 3, note that any pc-minor of G will be a subgraph of Q_2, thus outerplanar. Among all partial cubes of dimension 3, the only ones containing a subdivision of $K_{2,3}$ or K_4 are G_3 and Q_3. □

From now we can restrict to partial cubes of isometric dimension at least 4. We start with those containing only a subdivision of $K_{2,3}$.

Lemma 4.6. *Let G be a partial cube with $\dim(G) \geq 4$ containing a subdivision of $K_{2,3}$ but none of K_4 such that no pc-minor of G contains a subdivision of $K_{2,3}$. Then $G = L$.*

Proof. Among all subdivisions of $K_{2,3}$ contained in G, we choose a $K_{2,3}$-subdivision H contained in G with the minimum number of vertices. Let a, b, c, d, z be the *original vertices* of $K_{2,3}$, with $\deg_H(a) = 3 = \deg_H(z)$. H consists in three disjoint paths $\overline{abz}$, $\overline{acz}$, and $\overline{adz}$ called *main paths*. Each one of these paths contains at least two edges in two different Θ-classes. We can assume that b, c, d are the first vertex in each main path respectively, i.e., $ab, ac, ad \in E(H)$. Let E_1, E_2, E_3 be Θ-classes such that $ab \in E_1$, $ac \in E_2$, $ad \in E_3$.

Claim 4.7. Let P be a main path. Let $u, v \in P$ such that $\{u, v\} \neq \{a, z\}$. If $uv \notin E(H)$, then $uv \notin E(G)$.

Proof. Assume $uv \notin E(H)$ and $uv \in E(G)$. Since $u, v \in P$, there is a vertex $w \in P$ between u and v such that $w \notin Q := \overline{auvz}$. Since $\{u, v\} \neq \{a, z\}$, $\ell(Q) \geq 2$. Also, $w \notin Q$ implies that $\ell(Q) < \ell(P)$. Let H' be the graph built from H and replacing P for Q. H' is a subdivision of $K_{2,3}$ with less vertices than H, which is a contradiction to the fact that H is minimal in vertices (Figure 2). □

We conclude that there are no induced edges between vertices contained in the same main path, except maybe between a and z.

Claim 4.8. Let $u, v \in H$, vertices from two different main paths. Any path in G between u and v goes through a or z. In particular, $uv \notin E(G)$.

Proof. Let P, Q be two main paths such that $u \in P$, $v \in Q$. Let R be a path between u and v such that $a, z \notin R$. Note that $u \in P \backslash \{a, z\}$, $v \in Q \backslash \{a, z\}$ are two disjoint paths. Assume that $P \cap R = \{u\}$ and $Q \cap R = \{v\}$. Now a K_4-subdivision is formed, picking as original vertices a, u, v, z and six main paths, where R is one of them and the others paths are contained in H (Figure 2). □

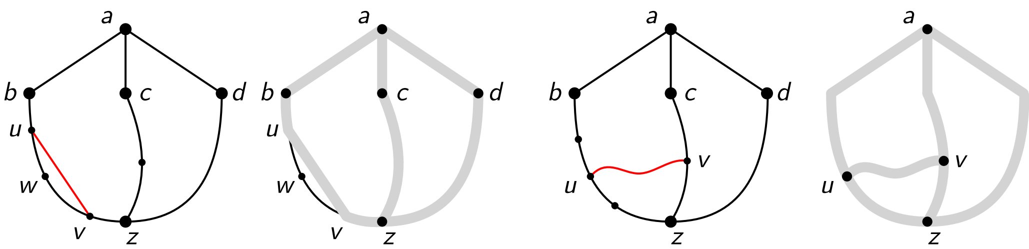

Figure 2: Claims 4.7 and 4.8: If uv exists, then: (left) there is a $K_{2,3}$-subdivision not containing w or (right) there is a K_4-subdivision.

Claims 4.7 and 4.8 imply that az will be (if it exists) the only edge in G induced by H.

Claim 4.9. a and z differ in only one coordinate, i.e., $az \in E(G)$.

Proof. Assume a and z differ in at least two coordinates, i.e., $a = (0, 0, \dots)$ and $z = (1, 1, \dots)$. Let E_1, E_2 be the Θ-classes corresponding to the first two coordinates. Since the three main paths are disjoint, there exist $e_{1b}, e_{1c}, e_{1d} \in E_1$ and $e_{2b}, e_{2c}, e_{2d} \in E_2$ such that $e_{1b}, e_{2b} \in \overline{abz}$, $e_{1c}, e_{2c} \in \overline{acz}$, $e_{1d}, e_{2d} \in \overline{adz}$. Then there exist three vertices $u_b \in \overline{abz}$, $u_c \in \overline{acz}$, $u_d \in \overline{adz}$ such that u_i is between e_{1i} and e_{2i} in each main path (Figure 3). Then each u_i has its first two coordinates either $(0, 1)$ or $(1, 0)$. In each eight combinations, at least two vertices have the same two first coordinates. Assume $u_b = (0, 1, \dots)$, $u_c = (0, 1, \dots)$. Now, let P be a short (u_b, u_c)-path. Any vertex $v \in P$ has got to have the same first two coordinates, i.e., $v = (0, 1, \dots)$. Then, neither a nor z can be in P. This is a contradiction with Claim 4.8. Then, a and z differ in only one coordinate, i.e., $az \in E(G)$. We can assume from now on that $az \in E_4$. □

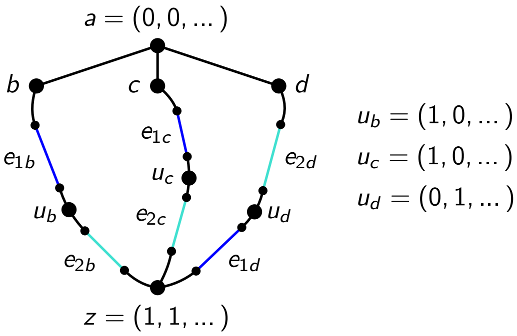

Figure 3: Claim 4.9: a short (u_b, u_c)-path cannot pass through a nor z.

Claim 4.10. Let P be a main path. Then, $\ell(P) = 3$ and $\Theta(P) = (E_i, E_4, E_i)$, where E_i is the Θ-class corresponding to the first edge of P starting from a, i.e., $i \in [3]$.

Proof. $P \cup \{az\}$ forms a cycle of length 4 or greater. Thus, this cycle has at least two edges in E_i and E_4. The other main paths Q, R already have three edges not contained in E_i. Then, $\pi_i(Q)$ and $\pi_i(R)$ do still have length greater than 2. Lemma 4.2 ensures that each Θ-class destroys H. Then, since E_i destroys H, we get $\ell(\pi_i(P)) < 2$. Thus, $\ell(\pi_i(P)) = 1$ and $\Theta(P) = (E_i, E_4, E_i)$. □

From Claim 4.10 we get to fully determine H. It turns out that $G[H] = H \cup \{az\} = L$.

Claim 4.11. $\dim(G) = 4$.

Proof. Thanks to Lemma 4.4, all Θ-classes have to contain an edge in $G[H]$, but $G[H] = L \subseteq Q_4$. □

Still, we have not fully determined $V(G)$ and there could be a vertex $v \in V(G) \backslash V(H)$.

Claim 4.12. $V(H) = V(G)$.

Proof. G is a partial cube, then G is connected. If $V(G)\backslash V(H) \neq \emptyset$, then there is a vertex $u \in V(G)\backslash V(H)$, adjacent to some $v \in V(H)\backslash\{a, z\}$. Assume $v = b$. Then either $bu \in E_2$ or $bu \in E_3$. Assume the first option. G is a partial cube implies $cu \in E(G)$ and $\Theta(cu) = E_1$. But that is a contradiction with Claim 4.8. Then $V(G) = V(H)$. □

Finally, $V(G) = V(H)$ and $G[H] = L$ imply that $G = L$, which finishes the proof of Lemma 4.6. □

Lemma 4.13. *Let G be a partial cube with* $\dim(G) = n \geq 4$ *containing a subdivision of K_4 such that no pc-minor of G contains a subdivision of K_f. Then $G = G_n$.*

Proof. Among all subdivisions of K_4 in G, we choose a subdivision H with the minimum number of vertices. Let a, b, c, d be the *original vertices* of K_4. The six edges of K_4 are called *main paths* in H. Let $e \in E(G[H])$. Then up to symmetry e has to be one of the following types (Figure 4):

(i) $e_1 = u_1v_1 \in E(H)$, $u_1, v_1 \in \{a, b, c, d\}$ are original vertices.

(ii) $e_2 = u_2v_2 \in E(H)$, $u_2 \in \{a, b, c, d\}$ is an original vertex and v_2 is a subdivision vertex of a main path containing u_2.

(iii) $e_3 = u_3v_3 \in E(H)$, u_3, v_3 are two subdivision vertices in the same main path.

(iv) $e_4 = u_4v_4 \notin E(H)$, $u_4 \in \{a, b, c, d\}$ is an original vertex and v_4 is a subdivision vertex of a main path that does not contain u_4.

(v) $e_5 = u_5v_5 \notin E(H)$, $u_5, v_5 \in \{a, b, c, d\}$ are original vertices.

(vi) $e_6 = u_6v_6 \notin E(H)$, $u_6 \in \{a, b, c, d\}$ is an original vertex and v_6 is a subdivision vertex of a main path containing u_6.

(vii) $e_7 = u_7v_7 \notin E(H)$, u_7, v_7 are two subdivision vertices of the same main path.

(viii) $e_8 = u_8v_8 \notin E(H)$, u_8, v_8 are two subdivision vertices of two adjacent main paths.

(ix) $e_9 = u_9v_9 \notin E(H)$, u_9, v_9 are two subdivision vertices of two opposite main paths.

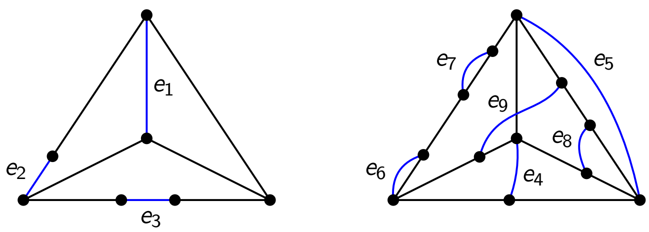

Figure 4: The nine different types of induced edges by H. On the left, the edges contained in H, on the right, the edges not contained in H.

Claim 4.14. Types (v), (vi), (vii), (viii), (ix) edges cannot exist (Figure 5).

Proof. **(v)** Assume $e_5 = ab \notin E(H)$. The main path $\overline{ab}$ cannot be a single edge. Thus, $\ell(\overline{ab}) \geq 2$. Then, there exists a vertex $w \in \overline{ab}$, $u \neq a, b$. Then, a K_4-subdivision H' is formed with the same original vertices a, b, c, d and the same main paths but replacing $\overline{awb}$ for the edge $e_5 = ab$. H' contains less vertices than H, contradiction.

(vi) Assume $e_6 = au \notin E(H)$. a and u are not adjacent in H. There is a vertex $w \in \overline{ab}$ between a and u. Then, there is a subdivision H' of K_4 with the same original vertices a, b, c, d and the same main paths but replacing $\overline{awub}$ for the path $\{au\} \cup \overline{ub}$. H' contains less vertices than H, contradiction.

(vii) Assume $e_7 = uv \notin E(H)$, $u, v \in \overline{ab}$. There exists a vertex $w \in \overline{ab}$ between u and v. Then there is another subdivision H' with the same original vertices a, b, c, d and the same main paths but replacing $\overline{auwvb}$ for the path $\overline{au} \cup \{uv\} \cup \overline{vb}$. H' contains less vertices than H, contradiction.

(viii) Assume $e_8 = uv \notin E(H)$, $u \in \overline{ab}$ and $v \in \overline{ac}$ are two subdivision vertices. There is a cycle going through a, u, v and at least a fourth vertex $w \in H$ (due to G being a partial cube). Assume $w \in \overline{au} \subset \overline{ab}$. Then there is another subdivision H' with original vertices v, b, c, d and the three main paths containing v being: $\overline{vd}, \overline{va} \cup \overline{ac}, \overline{bu} \cup \{uv\}$. H' contains less vertices than H, contradiction.

(ix) Even though we can find a subdivision of K_4 that has less vertices than H, there is another argument we can do. Assume $e_9 = uv$, $u \in \overline{ab}$, $v \in \overline{cd}$. Then, $H \cup \{uv\} = K_{3,3}$, where the bipartition is $V(K_{3,3}) = \{a, b, v\} \cup \{c, d, u\}$. That means G is not planar, which is a contradiction to Lemma 4.1. □

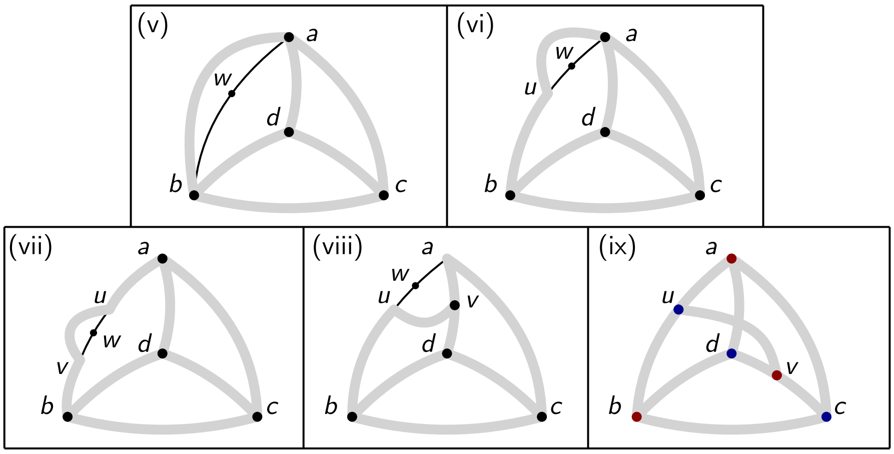

Figure 5: Representation of cases (v), (vi), (vii), (viii), (ix). In grey, the subdivisions of K_4 or $K_{3,3}$ deduced from the hypothesis of each case.

Now we have that $G[H] \backslash H$ can only have edges of type (iv), which are called *mixed edges*. Edges of type (i) are called *original edges* and edges of types (ii) and (iii) are called *subdivision edges*.

Claim 4.15. Let E_i be a Θ-class. E_i contains an original edge o mixed edge (types (i) o (iv)).

Proof. Thanks to Lemma 4.4, we know $E_i[H] := E_i \cap E(G[H]) \neq \emptyset$, since $G \in \Omega$ and H is a subdivision of K_4. Assume every edge in $E_i[H]$ is type (ii) or (iii), i.e., they are all subdivision edges. Contract all edges of $E_i \backslash E_i[H]$ (edges in E_i not induced by H). Lemma 4.3 implies $H = H/(E_i \backslash E_i[H])$, i.e., H is not affected by the contraction of $E_i \backslash E_i[H]$. Now, if we contract $E_i[H]$, we will have contracted all edges of E_i. Due to Lemma 4.2, $\pi_i(G)$ will not contain any subdivision of K_4. However, we are assuming all edges in $E_i[H]$ are subdivision edges, i.e., all edges in $E_i[H]$ are contained inside the main paths. There cannot be any main path containing only edges in E_i (except if the main path is a single edge, but in that case it would be an original edge). Then $\pi_i(H)$ still contains the same main paths contracted, but never until being fully contracted. Then, $\pi_i(G)$ contains $\pi_i(H)$ as a subgraph, which is still a subdivision of K_4. That is a contradiction which means that E_i has to have an original edge or a mixed edge (types (i) and (iv)). □

Claim 4.16. G contains at least one mixed edge (type (iv)).

Proof. We proof G cannot have more than three original edges. Since $n := \dim(G) \geq 4$, there is at least one Θ-class containing mixed edge. Assume E_1, E_2, E_3 are Θ-classes each one containing an original edge. Except symmetries, they can only form a C_3, P_3 or $K_{1,3}$ inside K_4. Let E_4 be a Θ-class. A fourth original edge in E_4 would form a C_4 or a $C_3 +_1 P_1$ together with the other three. A C_4 in a partial cube cannot have four different Θ-classes and a $C_3 +_1 P_1$ has an odd cycle, thus, E_4 cannot contain an original edge. Then, Claim 4.15 implies that E_4 necessarily contains a mixed edge. Moreover, G contains at least $n-3$ mixed edges. □

Claim 4.17. All mixed edges are incident to the same original vertex.

Proof. Let $e, f \in E(G)$ be two mixed edges incident to two different originals vertices. Assume $e = au$ and $f = dv$, $u, v \in V(H)$, being two subdivision vertices. Up to symmetries we have four cases; see Figure 6:

(i) $u, v \in \overline{bc}$.

(ii) $u \in \overline{bd}$ and $v \in \overline{bc}$.

(iii) $u \in \overline{bd}$ and $v \in \overline{ac}$.

(iv) $u \in \overline{bd}$ and $v \in \overline{ab}$.

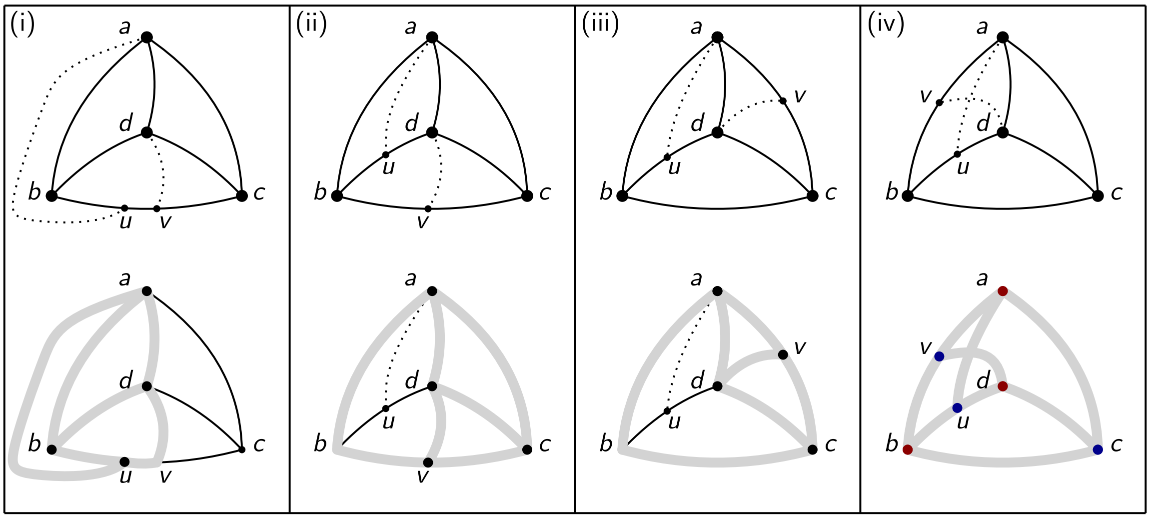

Figure 6: Cases (i), (ii), (iii), and (iv) of Claim 4.17.

In cases (i), (ii), (iii), just like in Figure 5 we can find a subdivision of K_4 with strictly less vertices than H. In case (iv) we can find a subdivision of $K_{3,3}$, contradicting that it is planar by Lemma 4.1. Hence, all mixed edges are incident to the same original vertex. □

We can assume all mixed edges are incident to d.

Claim 4.18. The main paths $\overline{ad}$, $\overline{bd}$, $\overline{cd}$ are indeed original edges, i.e., $ad, bd, cd \in E(H)$.

Proof. Claim 4.16 says there is at least a mixed edge $e \in E(G[H])$. Assume $e = du$, $u \in \overline{bc}$. There are three K_4-subdivisions H_1, H_2, H_3 taking as original vertices $\{b, c, d, u\}$, $\{a, c, d, u\}$, $\{a, b, d, u\}$, respectively. H having the minimum number of vertices implies $ad, bd, cd \in E(H)$. □

Now we know H contains three original edges and $n-3$ mixed edges. Thus, $\deg_G(d) = n$. We still need to know about the outer cycle of H, $Z := \overline{abca}$. From now on, we will not differentiate between the original vertices a, b, c and the other vertices in Z adjacent to d through a mixed edge. We will denote as $v_1, \ldots, v_n \in Z$ the vertices adjacent to d in G, ordered consecutively, and $E_1, \ldots, E_n$ the Θ-classes of edges $dv_1, \ldots, dv_n$, respectively. Analogously, we will not differentiate H from any other subdivision of K_4 taking d and any three vertices $v_i \in Z$, since they all have the same number of vertices (minimal, by hypothesis). $\forall i$, let $P_i := \overline{v_i v_{i+1}} \subseteq Z$ be the path not containing any other v_j (Figure 7(i)).

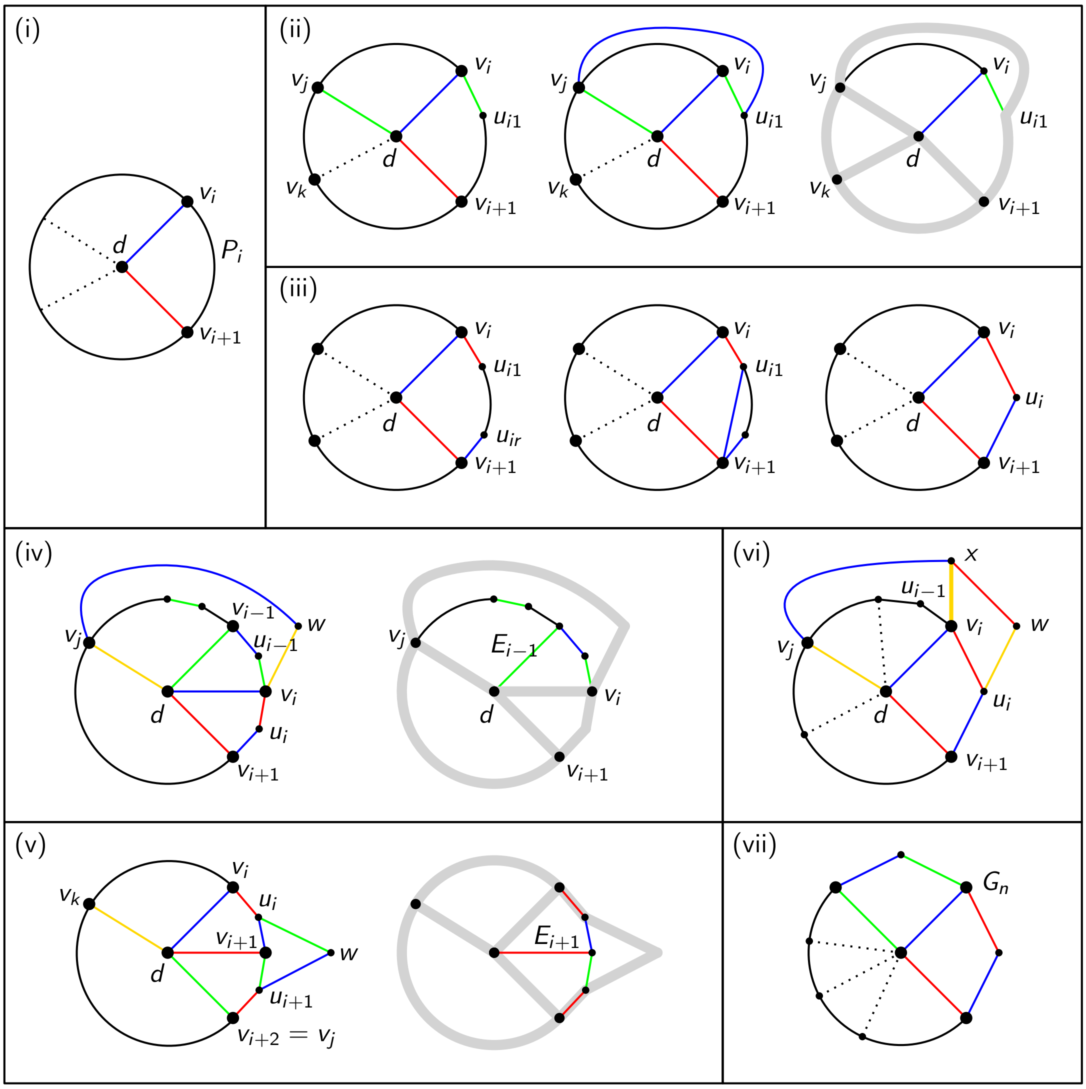

Figure 7: Summary of the different steps and cases in Claims 4.19 and 4.20.

Claim 4.19. $\text{Long}(P_i) = 2$ and $\Theta(P_i) = (E_{i+1}, E_i)$.

Proof. Assume $P_i = (v_i, u_{i1}, \dots, u_{ir}, v_{i+1})$. Keeping in mind that $n \geq 4$ and H is minimal in vertices, we can deduce that $v_i u_1 \in E_{i+1}$ and $u_r v_{i+1} \in E_i$ (Figure 7(ii)). Using partial cube properties, we deduce that $u_1 = u_r$. Thus, $\ell(P_i) = 2$ and $\Theta(P_i) = (E_{i+1}, E_i)$ (Figure 7(iii)). □

We deduce from Claim 4.19 that $Z = (v_1, u_1, v_2, u_2, \dots, v_n, u_n, v_1)$ and $G_n \subseteq G$. Moreover, we have that $G_n = G[H] = H \cup \{dv_i, 1 \leq i \leq n\}$.

Claim 4.20. $V(G) = V(G_n)$.

Proof. Assume $w \in V(G)\backslash V(H)$ is adjacent to a vertex $v \in V(H)$. v has to be in Z. We have two options:

(i) $v = v_i \in Z$, $1 \leq i \leq n$.

(ii) $v = u_i \in Z$, $1 \leq i \leq n$.

(i) Assume $wv_i \in E(G)$ and $wv_i \in E_j$. Note that $j \neq i-1, i, i+1$, and can exist since $n \geq 4$ (if $n = 4$, then there is only one option for j). To complete a square, we must have $wv_j \in E(G)$, $wv_j \in E_i$. Then there is a new K_4-subdivision with original vertices d, v_i, v_{i+1}, v_j, in which E_{i-1} does not contain any induced edge by H'. But this cannot happen, as Lemma 4.4 affirms that $E_{i-1}[H'] \neq \emptyset$ (Figure 7(iv)).

(ii) Assume $wu_i \in E(G)$, $wu_i \in E_j$. Note that $j \neq i, i+1$. We split it in two new cases for j:

(a) $j = i-1$ or $j = i+2$, i.e., v_j is consecutive to v_i or v_{i+1} in Z.

(b) $j \neq i-1, i+2$, i.e., v_j is not consecutive to v_i nor v_{i+1} in Z (cannot happen if $n = 4$).

(a) By symmetry, assume $j = i+2$. The square $\{w, u_i, v_{i+1}, u_{i+1}\}$ is completed, so we have $wu_{i+1} \in E_i$. But note that now G contains a K_4-subdivision H' with original vertices d, v_i, v_{i+2}, v_k, where k can exist since $n \geq 4$ (Figure 7(v)). Note that $|H'| = |H|$ and E_{i+1} does not contain any original or mixed edge in H', which contradicts Claim 4.15.

(b) Assume $j \neq i-1, i+2$. The path $(w, u_i, v_{i+1}, d, v_j)$ has length 4 and two edges in E_j. Then a short path $P = \overline{wv_j}$ has to have length 2, i.e., $P = (w, x, v_j)$, $x \notin Z$, and $\Theta(P) = (E_{i+1}, E_i)$. This forces the edge xv_i to exist and be contained in E_j. v_1x satisfies case (i) conditions, which we have already seen that it leads to a contradiction (see Figure 7(vi)).

Every case leads to absurdity. Then, there are no edges vw between $v \in Z$ and $w \in V(G)\backslash V(G_n)$. Since G is connected, we get $V(G)\backslash V(G_n) = \emptyset$, i.e., $V(G) = V(G_n)$. □

We have that $G[G_n] = G$ and $V(G_n) = V(G)$. Finally we conclude that $G = G_n$ (Figure 7(vii)), which finishes the proof of Lemma 4.13. □

4.3 Final results

Theorem 4.21. *The excluded pc-minors for outerplanar partial cubes are L, Q_3, and G_n for $n \geq 3$.*

Proof. Let G be a non-outerplanar partial cube such that every pc-minor of G is outerplanar. Chartrand–Harary [9] prove that non-outerplanar graphs contain $K_{2,3}$ or K_4 as a minor and it is easy to see that hence they contain a subdivision of $K_{2,3}$ or K_4. In particular this holds for G. By Lemmas 4.5, 4.6, and 4.13 we obtain that any pc-minor minimal non-outerplanar partial cubes must be a member of $\{L, Q_3, G_n, n \geq 3\}$. The proof that all elements of $\{L, Q_3, G_n, n \geq 3\}$ pc-minor minimal non-outerplanar partial cubes can be found in [26]. □

Since Lemmas 4.5, 4.6, and 4.13 are very specific concerning the graph that is obtained as a subdivision and series-parallel graphs are exactly those not containing a subdivision of K_4, see e.g. [8], we get:

Theorem 4.22. *The excluded pc-minors for series-parallel partial cubes are Q_3 and G_n for $n \geq 3$.*

5. Conclusions

The next natural minor-closed class are planar partial cubes, which have been characterized in different ways [1, 14]. Computer experiments show that in isometric dimensions 4, 5, 6 there are already $9 + 61 + 272 = 344$ pc-minor-minimal non-planar partial cubes. Considering pc-minor-minimal non-planar partial cubes such that all their isometric subgraphs are planar yields $2 + 10 + 34 = 46$ graphs. Looking only at pc-minor-minimal non-planar median graphs gives $1 + 4 + 8 = 13$ obstructions. Another possible class to attack are apex-outerplanar partial cubes, i.e., graphs that become outerplanar after removing some vertex. This minor-closed class lies between outerplanar and planar graphs, its 57 excluded minors are known; see [15]. For any excluded pc-minor G of outerplanar partial cubes, $G \square K_2$ is an excluded pc-minor of apex-outerplanar partial cubes as well as for planar partial cubes, i.e., in both cases the list is infinite.

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Societat Catalana de Matemàtiques

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SCM Master Thesis Day

Last October 6th we celebrated with a considerable attendance the first SCM Master Thesis Day. The proposal of this established day with intended annual periodicity is an initiative by the Catalan Society of Mathematics (SCM) which wants to give the opportunity for those who have just obtained a master's degree in mathematics at a university from a Catalan speaking area (Vives network) to present their Master's Final Thesis. It is also about giving recent master graduates the opportunity to participate and present their first communication in a scientific workshop. Another objective of this day is to publicise the Master of Mathematics in the universities of the Vives network and energize the community of young mathematicians that are starting their research career.

The activities of the day were held at the headquarters of the Institut d'Estudis Catalans (IEC) and had the participation as speakers of nine students. In addition, there was the presentation of the three master's theses awarded with the Evariste Galois 2023 prize (the winner and two accessits). The Evariste Galois prize is an award given by the SCM to the best final master's thesis of the previous year.

The conference had a scientific and organizing committee formed by Montserrat Alsina (president of the SCM), Josep Vives (vice-president of the SCM), Xavier Massaneda (coordinator of the UB Master's in Advanced Mathematics), Jordi Saludes (coordinator of the UPC Master's in Advanced Mathematics) and Xavier Bardina (editor-in-chief of *Reports@SCM*). This activity has been partially funded by the Secció de Ciències i Tecnologia of the IEC.

Reports@SCM collects in this issue the extended abstracts of ten of the twelve presentations of the day.

Extended Abstracts

REPORTS@SCM
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Convergence to the Brownian motion

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Resum *(CAT)*
El *moviment Brownià* és un procés estocàstic que modelitza el moviment de partícules suspeses en un líquid o gas. En matemàtiques, prén un rol vital en el càlcul estocàstic. En aquest treball es demostren tres resultats diferents de convergència cap a aquest. El primer resultat que es demostra és el teorema de Donsker. El segon resultat consisteix en demostrar la convergència en distribució d'uns processos estudiats per Daniel Stroock. L'últim resultat és la demostració de la convergència quasi segura d'uns processos anomenats Processos de Transport Uniforme, els quals són prèviament presentats.

Keywords: *Brownian motion, convergence, convergence in distribution, almost sure convergence.*

Abstract

During the 19th century, scientists started developing the discipline of statistical mechanics. Basically, they started treating physical systems mathematically. In 1859, James Clerk Maxwell presented a work on the kinetic theory of gases where he assumed that the gas particles move in random directions at random velocities. This was the starting point for the development of the statistical physics during the second half of the 19th century. During this period of time, Thorvald N. Thiele, in 1880, published a paper where he described the mathematics behind the *Brownian motion*.

The Brownian motion describes the random movement of particles suspended in a liquid or a gas. Despite it was first described by the botanist Robert Brown, in 1827, while he was looking at pollen particles through a microscope, it was not until the end of the 19th century that the mathematics behind this motion were addressed. Furthermore, it was not until 1900 when Louis Bachelier modeled for the first time, and under the supervision of Henri Poincaré, the stochastic process that we know as the Brownian motion.

This Brownian motion is the protagonist of this thesis. What we do is prove different results regarding some type of convergence towards this Brownian motion.

One of the results that we see is a classical result which is the Donsker's theorem. For this result we follow the first chapter, and part of the second one, of the book *Convergence of Probability Measures* by Patrick Billingsley [1]. To reinforce these two chapters we also follow the book *Curs de Probabilitats* by David Nualart and Marta Sanz [2].

In order to state and prove this result, we first discuss about different notions of convergence such as the weak convergence, the convergence in distribution or the convergence in probability.

Moreover, we discuss the concept of tightness, and also the notion of weak convergence, in the set of continuous functions on $[0, 1]$. To finish, we see the definitions of the Wiener measure and the Brownian motion.

With all this previous work, we are able to state the Donsker's theorem and prove that the stochastic processes that this theorem defines converge in distribution towards the Brownian motion.

Another result that we prove is the convergence in distribution of a particular type of stochastic processes. These processes that we define were presented by Mark Kac but was Daniel Stroock who explicitly proved their convergence. This result can be found in the work of Stroock, *Lectures on Topics in Stochastic Differential Equations* [3]. Even so, we follow a presentation made by Xavier Bardina on December 18, 2014 at Bucuresti called *On the Kac–Stroock Approximations* in order to achieve the proof of this result.

The last result that we see is the almost sure convergence of the uniform transport processes towards the Brownian motion. In order to prove this result, we follow the paper written by Richard J. Griego, David Heath and Alberto Ruiz-Moncayo, Almost sure convergence of uniform transport processes to Brownian motion [4].

We also use classical books as *An Introduction to Probability Theory and its Applications* by William Feller or *Studies in the Theory of Random Processes* by Anatoliy V. Skorokhod to complement the proof.

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Extended Abstracts

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Limit distribution of Hodge spectral exponents of plane curve singularities

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Resum *(CAT)*
K. Saito va formular la pregunta de si una distribució contínua és el límit de la distribució a l'interval $[0,1)$ dels exponents espectrals de Hodge d'una hipersuperfície, quan aquesta es mou en un sentit que s'ha de precisar. Ell ho va demostrar per a corbes planes irreductibles amb un límit molt específic. En aquest treball ens centrem en el cas de corbes planes irreductibles, per les quals explorem diferents formes d'assolir la distribució límit. També estudiem a on la distribució contínua és una cota superior per a la funció de distribució acumulada d'aquests invariants.

Keywords: *Hodge spectral exponents, plane curve singularities.*

Abstract

In the context of local Algebraic Geometry, we focus on the problem of understanding isolated singularities of complex hypersurfaces, described by an analytic equation $f \in \mathbb{C}\{x_0, \dots, x_n\}$ in a neighbourhood of the origin. In this work we study the distribution of a set of numerical invariants of f, the Hodge spectral exponents, for the particular case of irreducible plane curve singularities.

The Hodge spectral exponents $\alpha_1 \leqslant \cdots \leqslant \alpha_\mu$ are a set of μ rational numbers in the interval $(0, n+1) \subset \mathbb{R}$, where μ is the Milnor number and $\mathbb{C}^{n+1}$ is the ambient space of our hypersurface. This set is symmetric with respect to $\frac{n+1}{2}$ and is preserved by deformations (of the hypersurface) with constant Milnor number. The definition of the Hodge spectral exponents is based on the construction by Steenbrink of a mixed Hodge structure on the cohomology of the Milnor fiber. In addition, they are related to other major invariants of singularities such as the jumping numbers, which coincide with the Hodge spectral exponents in the interval $[0,1)$, and for plane curves both sets provide the same information on the singularity. Another related invariant is the geometric genus, which satisfies the relation $p_g = \#\{i | \alpha_i \leqslant 1\}$.

Kyoji Saito introduced the characteristic function $\chi_f(T)$ as the normalized spectrum, or equivalently as the Fourier transform of the (discrete) distribution of Hodge spectral exponents $D_f(s)$. By calculating the characteristic function he noticed that, for some sequences of hypersurfaces, the distribution of Hodge spectral exponents converges to a certain continuous distribution $N_{n+1}(s)$, which only depends on the dimension. Following this, K. Saito asked two main questions:

- For which sequences of hypersurfaces does the distribution of Hodge spectral exponents converge to $N_{n+1}(s)$?
- For which values $r \in \left(0, \frac{n+1}{2}\right) \subset \mathbb{R}$ is the cumulative distribution of $N_{n+1}(s)$ up to r an upper bound for the cumulative distribution of $D_f(s)$ up to r?

The main goal of this work is to provide partial answers to K. Saito's questions. We work within the case of irreducible plane curves, for which we have Morihiko Saito's explicit formula for the Hodge spectral exponents, and equivalently for the characteristic function. To have a better understanding of the problem, we elaborate on how the Fourier transform relates different concepts appearing in K. Saito's paper.

Regarding the convergence of the distribution of Hodge spectral exponents, K. Saito gave a partial result for irreducible plane curves, taking a very specific limit in terms of the last Puiseux pair of the curve. In this work we compute more general limits with respect to the Puiseux pairs. Consequently, we see that for some limits the distribution $D_f(s)$ does not converge to $N_2(s)$ but to other distributions.

With respect to the cumulative distribution, we prove a closed formula for $\phi_f(r)$, defined as the difference of the cumulative distributions for $N_2(s)$ and $D_f(s)$. This new tool allows us to give an alternative proof to a restricted version of a result by Tomari that states $\#\left\{i|\alpha_i \leqslant \frac{1}{2}\right\} < \frac{\mu}{8}$. In addition, it provides the means to bound $\phi_f(r)$ and thus determine intervals of values r where $\phi_f(r)$ is positive (or negative).

Moreover, thanks to this closed formula for $\phi_f(r)$, we are be able to calculate limits of the distribution of Hodge spectral exponents with a different procedure. This way we prove more general theorems on for which limits does the distribution $D_f(s)$ converge to $N_2(s)$, mainly in terms of the log-canonical threshold $\mathrm{lct}(f)$. For further reference see [1, 2, 3].

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Extension of ϕ-Lipschitz functions

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Resum *(CAT)*
Els teoremes d'extensió de McShane i Whitney poden generalitzar-se de diverses maneres, permetent estendre funcions des de subespais metrics a l'espai sencer preservant la constant de Lipschitz. En aquest sentit, proposem introduir una funció ϕ creixent, positiva i subaditiva que, en compondre-la amb una mètrica, s'obté una altra mètrica que millora l'extensió. A més, la noció d'espai d'índexs és introduïda i els resultats d'aquest treball generalitzen els ja coneguts sobre índexs de Lipschitz per al cas dels ϕ-Lipschitz.

Keywords: *Lipschitz function, extension, index.*

Abstract

The classical Lipschitz extension theorems for real functions, due to McShane and Whitney, have found numerous applications in various fields: economics, social sciences and, more recently, in the field of artificial intelligence. These results can be generalised in various ways, by extending the class of functions to which they can be applied, or by weakening the metric conditions. In all these cases, we can extend functions defined on metric subspaces to the simple space, preserving the Lipschitz constant. In this sense, the proposal of this work consists in introducing an increasing, positive and subadditive function ϕ which, when composed with a metric, obtains another function with similar properties to the original metric.

Furthermore, in order to provide a functional basis for the recent interest in numerical indices in various disciplines (stock markets, forecasting, demography, etc.), the notion of index space is introduced. These indices are real Lipschitz functions which, depending on the problem, may satisfy additional conditions such as the Katetov condition. The results of this work generalise the already known results on Lipschitz indices for the case of ϕ-Lipschitz, in addition to studying the compactness of the set of corresponding standard indices. The properties of the approximation that make it possible to work with this functional basis to design artificial intelligence algorithms in ϕ metric models are also presented.

Finally, this paper presents and contextualises a problem of recent interest related to urban liveability indices. We will see how modelling through index expansions and their respective extensions will be useful, and will serve as an example to compare the methodology we have developed with the original one. The results obtained show that, in general, the extension process is improved by composing the original metric with a ϕ function, on the nature of which the final result will depend. For further reference see [1, 2, 3, 4].

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Extended Abstracts

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Counting subgroups using Stallings automata and generalisations

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Resum *(CAT)*
El problema de comptar els subgrups d'índex finit del grup lliure va ser abordat el 1949 per Marshall Hall, que va proporcionar una fórmula recursiva per al nombre de subgrups d'un índex finit donat en un grup lliure de rang finit. Aquest treball proporciona una prova del resultat de Hall utilitzant la teoria dels autòmats de Stallings. A més, veurem com obtenir una fórmula similar en el cas dels grups lliure per lliure-abelians, fent servir una generalització de la teoria dels autòmats de Stallings per a la família de grups lliure per lliure-abelians.

Keywords: *Stallings automata, enriched automata, finite index subgroup.*

Abstract

It is known that, in a finitely generated group, there are finitely many subgroups of a fixed finite index. It is therefore very natural to ask how many subgroups of a certain finite index there are in a group. In 1949, Marshall Hall Jr. answered this question in the case of free groups, providing a recursive formula for the number of subgroups of a given finite index in $\mathbb{F}_n$, the free group of rank n. In this work, we show how Hall's result can be understood and proved using the theory of Stallings automata. In addition, we present a recently developed generalisation of the theory of Stallings automata to the case of free times free-abelian groups and apply it to obtain a formula for the number of subgroups of a given finite index in $\mathbb{F}_n \times \mathbb{Z}^m$.

Since free times free-abelian groups are the direct product of a free group and a free-abelian group, we start by studying the problem of counting finite index subgroups in each of the factors separately.

In the free-abelian case, one can obtain a recursive formula for the number of subgroups of a given finite index in $\mathbb{Z}^m$ applying techniques similar to those of linear algebra. Essentially, a bijection is established between the subgroups of index k of $\mathbb{Z}^m$ and certain kind of matrices with integer coefficients and determinant equal to k. Using a recursive argument to count these matrices, the desired formula is obtained.

In the case of the free group, the corresponding formula was obtained by M. Hall in 1949 and it is possible to reformulate his proof using the celebrated theory of Stallings automata.

In 1983, Stallings established the basis for the study of the subgroups of the free group by means of a graphical representation consisting in a certain type of directed labelled graphs which are now known as Stallings automata. In this graphical representation, the elements of the subgroup correspond with the labels of certain closed walks in the Stallings automata. The key of this theory is the obtention of a bijection between subgroups of the free group and Stallings automata. The techniques of the theory of Stallings automata have allowed to solve in the free group many of the algorithmic problems which are usually posed in group theory (for example, the finite index problem, the intersection problem or the membership problem).

In particular, to arrive at Hall's formula, one can exhibit a map from the set of n-tuples of k permutations to the set of Stallings automata whose arcs are labelled with n elements and which are saturated and have k vertices (these objects are in bijection with the subgroups of $\mathbb{F}_n$ that have index k). Analysing the cardinal of the fibers of this map, one can deduce Hall's formula.

The classical theory of Stallings has been extended to subgroups of free times free-abelian groups. In this case, one can establish a bijection between subgroups of $\mathbb{F}_n \times \mathbb{Z}^m$ and certain type of automata enriched with abelian labels that encode the information corresponding to the abelian part of these groups. As an application of this bijection, we give a geometric characterisation of the subgroups of a given finite index in $\mathbb{F}_n \times \mathbb{Z}^m$ and, combining this characterisation with the existing formulae for the free and free-abelian cases, we obtain a new formula for the number of subgroups of a given finite index in $\mathbb{F}_n \times \mathbb{Z}^m$. For further reference see [1, 2, 3, 4].

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The Gromov–Hausdorff distance between compact metric spaces

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Resum *(CAT)*
Aquest treball proporciona una introducció a la distància de Gromov–Hausdorff, discutim la seva definició original i la seva relació amb les correspondències entre espais. Demostrem que la distància de Gromov–Hausdorff serveix com a mètrica per al conjunt de classes d'isometria d'espais mètrics compactes. Els objectius principals d'aquest estudi són establir l'existència d'una pseudomètrica en la unió disjunta de X amb Y que aconsegueix la distància de Gromov–Hausdorff entre espais compactes X i Y, i per establir límits per al Gromov–Hausdorff distància entre esferes de diferents dimensions.

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Keywords: *Hausdorff, metric, correspondance.*

Abstract

The Gromov–Hausdorff distance between metric spaces X and Y, denoted by $d_{GH}(X, Y)$, quantifies the extent to which X and Y fail to be isometric. The Gromov–Hausdorff distance is used in many areas of geometry, in applications to shape and data comparison/classification, one aims to estimate either the Gromov–Hausdorff distance between spaces or the Gromov–Wasserstein distance, which is one of its optimal transport induced variants.

Let A, B be pseudo-metric spaces. The *Gromov–Hausdorff distance* (see [2]) between A and B, denoted by $d_{GH}(A, B)$, is the infimum of all $\varepsilon \geq 0$ so that there is a pseudo-metric space M and isometric embeddings $i_A\colon A \to M$ and $i_B\colon B \to M$ such that $d_M(i_A(A), i_B(B)) \leq \varepsilon$, where d_M denotes Hausdorff distance in M. Then we prove that we can actually restrict ourselves to pseudo-metrics on the disjoint union of A and B.

We introduce correspondences between sets and the concept of distortion of a correspondence in order to prove that the Gromov–Hausdorff distance can be computed using them. For any two pseudo-metric spaces X and Y,

$$d_{GH}(X, Y) = \frac{1}{2} \inf_C \{\mathrm{dis}(C)\},$$

where the infimum is taken over all correspondences C between X and Y. The set of isometry classes of compact metric spaces endowed with the Gromov–Hausdorff distance is a metric space.

We study the structure of the metric space of metrics on a given set. We focus on the case where the given space is a complete and compact metric space. Then we study the set of closed relations and the subset of closed correspondences (see [3]), which turns out to be a compact set. We prove that the distortion function is a continuous function. Hence we obtain the following result: For any two compact metric spaces X and Y there exists a correspondence R such that $d_{GH}(X, Y) = \frac{1}{2}\operatorname{dis}(R)$.

We focus on the case of estimating Gromov–Hausdorff distances between spheres of different dimensions (see [1, 5], for a generalization see [4]). We relate Gromov–Hausdorff distance, Borsuk–Ulam theorems, and Vietoris–Rips complexes as follows. Estimating the Gromov–Haudorff distance $d_{GH}(X, Y)$ for metric spaces X and Y involves bounding the distortion of a function $f\colon X \to Y$, which measures the extent to which f fails to preserve distances; the more functions between X and Y distort the metrics, the larger $d_{GH}(X, Y)$ must be. When X and Y are spheres, it is sufficient to consider odd functions. We transform an odd function $f: \mathbb{S}^k \to \mathbb{S}^n$ into a continuous odd map between Vietoris–Rips complexes. Then we obstruct the existence of such maps with the $\mathbb{Z}/2$ equivariant topology of Vietoris–Rips complexes, measured via the following quantity: For $k \geq n$, we define

$$c_{n,k} = \inf\{r \geq 0 \mid \text{there exists an odd map } \mathbb{S}^k \to VR(\mathbb{S}^n; r)\}.$$

Due to a theorem of Hausmann, there is a homotopy equivalence $VR(\mathbb{S}^n; r) \simeq \mathbb{S}^n$ for sufficiently small r, and moreover there is an odd map $f\colon VR(\mathbb{S}^n; r) \to \mathbb{S}^n$. The Borsuk–Ulam theorem then implies that no odd map $\mathbb{S}^k \to VR(\mathbb{S}^n; r)$ exists for such r unless $k \leq n$. In particular, $c_{n,n} = 0$. Therefore, the quantity $c_{n,k}$ represents the amount by which $\mathbb{S}^n$ needs to be "thickened" until it admits an odd map from $\mathbb{S}^k$.

We find bounds for the Gromov–Hausdorff distance between spheres: For all $k \geq n$, the following inequalities hold:

$$2 \cdot d_{GH}(\mathbb{S}^n, \mathbb{S}^k) \geq \inf\{\operatorname{dis}(f) \mid f\colon \mathbb{S}^k \to \mathbb{S}^n \text{ is odd}\} \geq c_{n,k}.$$

And that for every $n \geq 1$, we have that $d_{GH}(\mathbb{S}^n, \mathbb{S}^{n+1}) \leq \pi/3$.

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Extended Abstracts

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Poisson cohomology: old and new

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Resum *(CAT)*
Les varietats de Poisson tenen naturalment associat un complex de cocadenes, la cohomologia del qual es diu *cohomologia de Poisson*. En aquest treball estenem les tècniques de Guillemin, Miranda i Pires per calcular la cohomologia de Poisson de varietats b^m-Poisson. La cohomologia resultant té dimensió infinita i ve parametritzada per la foliació simplèctica del lloc singular Z. El fet que puguem mesurar la diferència entre la cohomologia de Poisson i la cohomologia de l'algebroide subjacent obre una porta a l'estudi de la cohomologia de Poisson de varietats més generals.

Keywords: *Poisson cohomology, b^m-symplectic geometry, deformation quantization, deformation theory.*

Abstract

Poisson geometry is a vast generalization of the Hamiltonian setting of classical mechanics, described abstractly by symplectic geometry. The deformation of Poisson structures is related to the theory of normal forms and to the problem of their classification. The deformation theory of a Poisson manifold (M, Π) is governed by the cohomology of the Lichnerowicz complex $(\mathfrak{X}^\bullet, \mathrm{d}_\Pi)$, called Poisson cohomology.

For an important class of Poisson structures, the singularities can be encoded in a suitable vector bundle E and analyzed by means of geometric techniques. Such Poisson structures Π are obtained from a symplectic Lie algebroid (E, ρ, ω) by pushforward under the anchor map. These structures were already introduced by Nest and Tsygan [3] in the study of deformation quantization of Poisson manifolds. Their differential and symplectic geometry was thoroughly investigated by Miranda and Scott [2]. A notable instance of this definition is b-symplectic geometry, where $E = {}^b\mathrm{T}M$ is called the b-tangent bundle. Sections of ${}^b\mathrm{T}M$ can be identified with smooth vector fields $X \in \mathfrak{X}(M)$ tangent to an embedded hypersurface $Z \subset M$. A generalization of this setting is b^m-symplectic geometry, where vector fields are assumed to be tangent to Z at least of order m. The work of Scott [4] laid the rudiments of such geometries.

The systematic use of these techniques in the study of Poisson cohomology was pioneered by Guillemin, Miranda and Pires [1] in the case of b-geometry. The complex of sections of ${}^b\mathrm{T}M$ is naturally included in the complex of smooth multi-vector fields and admits a restriction of the operator d_Π. Consequently, the cohomology of this sub-complex, called *b-Poisson cohomology*, is a subspace of the standard Poisson cohomology groups. A comparison lemma due to Mărcuţ and Osorno shows the inclusion morphism induces an isomorphism at the level of cohomology.

In this master thesis we expand on these techniques to answer the following question posed by Alan Weinstein: is the inclusion morphism an isomorphism in b^m-symplectic geometry? We prove the quotient complex $\mathfrak{X}^\bullet_Q(M) = \mathfrak{X}^\bullet(M)/{}^{b^m}\mathfrak{X}^\bullet(M)$, which measures the obstruction to the inclusion being an isomorphism in cohomology, has infinite-dimensional cohomology groups. By showing this complex can be localized to the degeneracy locus $Z \subseteq M$ of Π, we are able to use normal form theory to completely describe the cohomology groups of $\mathfrak{X}^\bullet_Q(M)$ and, using a long exact sequence, to ultimately recover the following expression for the Poisson cohomology of a general b^m-Poisson manifold:

$$\mathrm{H}^k_\Pi(M) \simeq \mathrm{H}^k(M) \oplus \mathrm{H}^{k-1}(Z) \oplus \left(\mathrm{H}^{k-1}_\Pi(\mathcal{F}_Z)\right)^{m-1} \oplus \left(\frac{\mathrm{H}^{k-1}(Z)}{\alpha \wedge \mathrm{H}^{k-1}(\mathcal{F}_Z)}\right)^{m-1}.$$

These results open several questions for future projects. For simple b^m-Poisson manifolds, where the computation of Poisson cohomology can be carried out explicitly, we have been able to measure the failure of the isomorphism in cohomology by appropriately unfolding the Poisson structure in terms of a real parameter ε. The integrating coboundaries in cohomology blow-up taking the limit $\varepsilon \to 0$, showing the analytic behaviour of the pathology. Can similar computations be carried out in general? Additionally, this example shows the cohomology of the symplectic algebroid might be very different from the cohomology of the resulting Poisson manifold. An interesting problem is to find general relations between both cohomologies in terms of the anchor map ρ.

Acknowledgements

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Long Density Parity Check codes

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Resum *(CAT)*
Els codis de Low Density Parity Check (LDPC), o codis de comprovació de paritat de baixa densitat, van ser proposats per Robert Gallager al 1963 a la seva tesi doctoral [3]. Tot i que la tesi demostra que existeixen LDPC asimptòticament òptims, es van abandonar pel seu cost computacional. Gràcies a la teoria de grafs expansors, sabem que poden ser codificats i descodificats en temps lineal. Aquest TFM [4] repassa la història dels LDPC i en presenta una nova família creada a partir d'estructures d'incidència anomenades Quadrangles Generalitzats.

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Keywords: *error correcting codes, expander graphs, communication theory.*

Abstract

To send information from a transmitter to a receiver through a noisy channel, the data is sent with some redundancy in order to correct the errors that might occur. We will assume in this work that the data to be sent is a sequence of bits. The transmitter sends blocks of n bits, k of which are data bits, and the remaining $n - k$ are redundancy bits. The way we map vectors of k data bits to vectors of n transmitted bits is called a *(binary) code*. The vectors of n transmitted bits of a given code are called its *codewords*. The fraction of data bits per codeword, k/n, is called the code's *rate*.

A receiver, upon receiving a vector of n bits (also known as a word), has to decide which vector of k data bits encoded this word. This is called decoding a word. To make the receiver's job easier, codes are designed in such a way that different codewords do not resemble each other. Thus, if some errors occur in the channel, the receiver can still distinguish which codeword was sent and perform the decoding successfully. Given two vectors, the *Hamming distance* between them is the number of coordinates where the vectors differ. Thus, if the *minimum Hamming distance* between codewords is large enough, when we decode the received word we can correct some errors that were introduced in the noisy channel.

Given a family of codes, we say that they are *asymptotically good* if their rate is bounded by a constant larger than zero, and their minimum Hamming distance grows linearly with block length (n). In 1948, Shannon used the probabilistic method to show that asymptotically good codes exist [5]. However, his method did not give explicit examples on how to obtain them. Moreover, these codes might not be easily encodable or decodable.

Error correcting codes need to be practical, which means that encoding and decoding must be cheap both in computation and storage. The most common solution is to use *linear codes*, which are characterized by the property that any linear combination of codewords is also a codeword. After Shannon's landmark paper, the race to find asymptotically good linear codes began. During the sixties and seventies, algebraic constructions proved that such codes exist, but they were not encodable and decodable in linear time.

In 1963, Gallager discovered Low Density Parity Check (LDPC) codes [3], which he found experimentally to have good performance and had a link to graph theory through random graphs. However, Gallager lacked the tools to give explicit arguments of all the good properties of these codes, namely the concept of *expander graphs*. These codes were somewhat forgotten, since the thought at the time was that they were not practical due to the computing power they required.

In the seventies, the concept of expander graphs was introduced, which allowed Tanner, Sipser and Spielman [8, 6] to produce stronger results than the ones Gallager had obtained with random graphs. These led to Spielman's discovery in 1996 of the first family of asymptotically good, explicit codes, with encoding and decoding time linear in block length [7], which he coined *expander codes*.

Nowadays, LDPC codes are extensively used. Most notably, they appear in Digital Video Broadcasting, Wi-Fi and 5G standards [2, 1]. They are also widely employed for various storage system applications. While Spielman's decoding algorithm gives stronger analytical results, usually variants of Gallager's probabilistic decoding method are used, as the latter has better performance in practice.

The work [4] is an introduction to LDPC codes, both in theory and practice. It also presents a new family of LDPC codes constructed from the point-line incidence structure of Generalized Quadrangles. These codes are quasi-cyclic, which means that their parity-check matrix can be represented as a series of cyclic permutations of a smaller base matrix. This structure aids in efficient hardware implementation of decoding algorithms. Numerical experiments show that GQ-LDPC codes out-perform random codes.

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Jet transport for General Linear methods

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Resum *(CAT)*
Estudiem la tècnica computacional anomenada "jet transport" per a la família d'integradors numèrics coneguts com a mètodes Generals Lineals (GLM), que generalitzen els reconeguts mètodes multipas lineals (LMM) i Runge–Kutta (RK). El jet transport és l'aplicació de l'aritmètica de sèries de potències truncades a un integrador numèric per tal d'obtenir la solució de les equacions variacionals (EV); és a dir, les equacions diferencials lineals que compleixen les derivades de la solució d'un problema de valor inicial (PVI). En particular, es discuteix la seva implementació i aplicacions.

Keywords: *numerical integration, variational equations, stiff problems.*

Abstract

The main subject of this thesis is to discuss how to apply the computational technique called jet transport to the family of numerical integrators known as General Linear methods. We also give instructions for its correct computational implementation, present numerical examples and applications. The content is organized into three chapters:

In the first chapter, we introduce General Linear methods (GLM), which are used for the numerical integration of initial value problems (IVP)

$$y'(x) = f(y(x)), \quad y(x_0) = y_0.$$

The formulation of an s-stage r-step General Linear method, for certain coefficient matrices $\mathbf{A} = [a_{ij}] \in \mathbb{R}^{s\times s}$, $\mathbf{U} = [u_{ij}] \in \mathbb{R}^{s\times r}$, $\mathbf{B} = [b_{ij}] \in \mathbb{R}^{r\times s}$, $\mathbf{V} = [v_{ij}] \in \mathbb{R}^{r\times r}$, is the following

$$Y_i^{[n]} = h\sum_{j=1}^{s} a_{ij} f(Y_j^{[n]}) + \sum_{j=1}^{r} u_{ij} y_j^{[n-1]}, \quad i = 1, \dots, s,$$

$$y_i^{[n]} = h\sum_{j=1}^{s} b_{ij} f(Y_j^{[n]}) + \sum_{j=1}^{r} v_{ij} y_j^{[n-1]}, \quad i = 1, \dots, r.$$

They are a natural generalization of the well-known Runge–Kutta (RK) and Linear Multistep methods (LMM), and thus they use the information of several previous steps as well as several stages (additional computations per step). Throughout the chapter, we study the properties of local error, order, convergence, stability, consistency and the linear stability for the three families (i.e., LMM, RK and GLM) and observe that those of GLM generalize those of LMM and RK methods. For further reference, see [1] and [3].

In the second chapter, we introduce jet transport for computing the numerical solution of the variational equations (VE); i.e., the linear differential equations that are satisfied by the derivatives (up to any order, with respect to the initial conditions) of the solution of an initial value problem (IVP). Denoting the solution of the IVP as $y(x; x_0, y_0)$ and its derivative with respect to the initial conditions as $V(x) := D_{y_0} y(x; x_0, y_0)$, the first order variational equations are

$$\begin{aligned} y'(x) &= f(x, y(x)), & y(x_0) &= y_0, \\ V'(x) &= D_y f(x, y(x)) V(x), & V(x_0) &= I. \end{aligned}$$

Jet transport is the application to a numerical integrator of the technique called automatic differentiation, which is based on the observation that the jet (set of derivatives) of a multivariate function is codified by its Taylor expansion, so that high order derivatives of a function can be computed by using the arithmetic of truncated power series, which can be implemented in a computer. Consequently, jet transport can be understood as the application of the arithmetic of truncated power series to a numerical integrator in order to obtain the solution of variational equations. For further reference, see [2] and [4].

The rest of the second chapter is devoted to the presentation of our two main contributions. First, we prove that the numerical integration with a GLM of an IVP with jet transport of any order is equivalent to the numerical integration with the same GLM of the VE of the same order. Second, we derive the expressions that the coefficients of the jets must satisfy for them to be solutions of implicit systems. This allows jet transport to be applied to implicit General Linear methods; that is, those GLM that have an implicitly defined integration step, and which are of great utility in solving the so-called “stiff” problems.

The third chapter concludes the project by discussing the implementation and applications of the contents developed in the previous chapters. Given the complexity of GLM, we limit ourselves to presenting an efficient implementation of Runge–Kutta methods (both explicit and implicit) with jet transport. To show its applications, we use this implementation to determine the periodic orbit and the period of the van der Pol problem (which depends on a parameter that increases the stiffness of the problem). We also compute the power expansion of the Poincaré map of the periodic orbit with respect to the parameter.

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This is joint work with Àngel Jorba and Joan Gimeno.

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Cayley graphs and endomorphism monoids

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Resum *(CAT)*
El tema principal del treball és l'estudi de diverses famílies especials de grafs simples com a grafs de Cayley de monoides i semigrups (és a dir, que es poden obtenir a partir de grafs de Cayley traient-ne els arcs múltiples, les direccions i els loops), tot seguint la línia d'algunes preguntes presents al *paper On monoid graphs* de K. Knauer i G. Puig i Surroca.

A nivell estructural, el projecte intenta arribar a aquestes preguntes de manera natural, progressant des de propietats bàsiques sobre grafs i semigrups a alguns petits resultats originals.

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Keywords: *algebraic graph theory, monoid graphs, semigroup theory*

Abstract

The main objective of this work was to study some families of simple graphs with special properties as *monoid* or *semigroup graphs*, that is, proving under which conditions they could be seen as the *underlying simple graph* of a directed graph built via the Cayley graph construction.

More specifically, we introduce and talk about the basic tools of the field of algebraic graph theory (we discuss the basic properties of the aforementioned Cayley graph construction, and a fairly young generalization of it by Yongwen Zhu as seen in [4]), introduce some recent interesting results in the literature by many authors, mostly by K. Knauer and coauthors (as in references [1], [2], [3]) and try to put together a comprehensive guide to try and understand the main difficulties and ideas used in one of the main lines of work in the field.

These motivations can be observed in our in-depth study of some families of outerplanar graphs as monoid graphs, which culminates in a small original characterization of outerplanar monoid graphs that admit a representation of the form $\underline{\mathrm{Cay}}\{M, \{a, a^2\}\}$ for a monoid M and an element $a \in M$; or our brief study of $K_4 \sqcup C_5$ as a non monoid but possibly semigroup graph.

Both of the main lines of original work presented in the project were originally motivated by the work of K. Knauer and Puig i Surroca, as an attempt to answer part of the Questions 6.2 and 6.3 they posed in [3].

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Matlis duality, inverse systems and classification of Artin algebras

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Resum *(CAT)*
Amb l'objectiu d'estudiar la classificació d'algunes famílies d'àlgebres artinianes, estudiarem alguns resultats importants sobre mòduls injectius i la dualitat de Matlis. Més específicament, estudiarem la correspondència de Macaulay, la dualitat de Matlis quan $R = k[[x_1, \dots, x_n]]$ amb l'ideal maximal $m = (x_1, \dots, x_n)$. Utilitzant aquesta, podem introduir-nos en les funcions de Hilbert, essencials en la classificació d'àlgebres artinianes. Tots aquests resultats han anat acompanyats de càlculs i exemples fets amb SINGULAR [3] i la llibreria INVERSE-SYST.LIB [4] del Dr. Joan Elias.

Keywords: *Matlis duality, inverse system, Artin algebra,* SINGULAR.

Abstract

Classifying Artin algebras could be a huge challenge in some cases. Luckily, we could use some theorems and results which make that process easier. We are talking about Matlis duality which will be defined later. Should one want to understand such duality, injective modules and other results should be studied previously. After that, Macaulay's correspondence, a particular case of Matlis duality, will be defined too.

As it has been said before, injective modules are the starting point. Let R be a commutative ring and E an injective R-module. By definition, we say that E is injective if and only if, for all injective morphism $i\colon A \to B$ and for all morphism $f\colon A \to E$, where A and B are R-modules, a morphism $g\colon B \to E$ exists such that all commute. Now, the existence of injective hulls of modules could be proven, essential for deducing the existence of minimal injective resolutions of R-modules.

In addition, when R is a Noetherian ring, we can define what a Bass number is and the relation between those and the minimal injective resolution of a finite R-module.

Now, it is time to define the Matlis duality. It ensures an isomorphism between Artin and Noetherian modules. Given A an R-module, let $(R, \mathfrak{a}, \mathbf{k})$ Noetherian local ring, its dual will be $A^\vee = \mathrm{Hom}_R(A, E)$, E an injective hull of $\mathbf{k}$, by Matlis duality. It was written and proved by Eben Matlis in 1958.

It is important to shed light on the particular case of Matlis duality: Macaulay's correspondence. In that scenario,

$$R = \mathbf{k}[[x_1, \dots, x_n]],$$

the ring of formal series, with maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)$. Studying this correspondence will be a useful tool. Not only will it be relevant when approaching the relation between Artin rings and Gorenstein rings, but also studying level rings. Level rings are the starting point when defining Irrabino's Q-decomposition of the associated graded ring of an Artinian s-level local k-algebra.

Macaulay's correspondence could be used to study how to achieve isomorphism classes of local algebras. Furthermore, an important result is reached, an isomorphism between two Artinian s-level algebras is defined by a matrix using Macaulay's inverse system.

Through all this project, some computations have been done with SINGULAR and INVERSE-SYST.LIB by J. Elias. As one of the main and last results, one can prove the existence of an isomorphism between some models for A and its inverse system when A is an Artin Gorenstein local $\mathbf{k}$-algebra with Hilbert function $\mathrm{HF}_A = \{1, 3, 3, 1\}$. Had it not been for SINGULAR, this proof would have been large and tedious.

Acknowledgements

I want to thank my supervisor Dr. Joan Elias for all his help and the way he guided me through all this thesis.

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